Neighbour-swap Graphs

Generating linear extensions of posets by adjacent transpositions

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Outline

Introduction

Set case

- Take a set: $\{1, 2, 3, 4\}$.
- Consider a permutation, like 1234.
- Perform a neighbour-swap, i.e. an adjacent transposition.
  - For example, we can swap the 1 and 2 to obtain 2134.
- Can one perform these neighbour-swaps (not necessarily starting with 1234) in such a way that every permutation is encountered exactly once?
- Steinhaus-Johnson-Trotter algorithm.

Multiset case

- Take a multiset: $\{1, 1, 2, 2\}$.
- Can one still perform neighbour-swaps in such a way that every permutation is encountered exactly once?
- We can visualize our problem using a neighbour-swap graph:

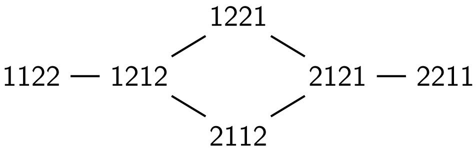


Figure: Neighbour-swap graph of the multiset $\{1, 1, 2, 2\}$.

- So, when is it possible for multisets?

Poset case

- There is an even more general way to create neighbour-swap graphs using posets.
- The poset on $\{1, 2, 3, 4\}$ with the order that 1 precedes 2 and 3 precedes 4 acts exactly like the multiset $\{1, 1, 2, 2\}$:

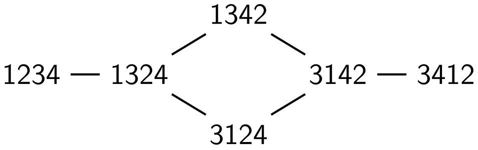


Figure: Neighbour-swap graph where 1 precedes 2 and 3 precedes 4.

Poset case

- The poset where 1 and 2 must both precede 3 and 4 gives a neighbour-swap graph that is unobtainable using just sets or multisets:

```
1234 — 2134
 |      |
1243 — 2143
```

Figure: Neighbour-swap graph where 1 and 2 must both precede 3 and 4.

Preliminaries

Poset definition

Definition

In our setting a poset P is defined as a *finite* set, also called the ground plane, combined with a relation $\prec$ which is irreflexive and transitive.

Example

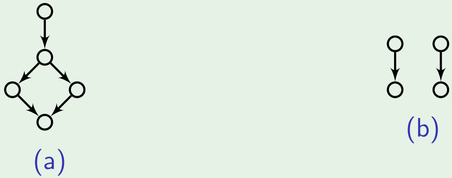


Figure: Hasse diagrams of two posets

Neighbour-swap graphs

Definition

Let $\mathscr{L}(P)$ denote the set of all linear extensions of a poset P.

Definition

A neighbour-swap graph $\mathscr{G}(P)$ of a poset P is defined as the graph on $\mathscr{L}(P)$ with edges corresponding to neighbour-swaps.

Theorem

Every neighbour-swap graph is bipartite where the two parts consist of the even and odd linear extensions.

Definition

Define the surplus $\mathscr{D}(\mathscr{G}(P))$, or $\mathscr{D}(P)$, as the absolute difference between the cardinality of the two parts of the bipartition.

Theorem

1. *If $\mathscr{G}(P)$ has a Hamiltonian path then $\mathscr{D}(P) \leq 1$.*
2. *If $\mathscr{G}(P)$ has a Hamiltonian cycle then $\mathscr{D}(P) = 0$.*

Proof.

Any path in $\mathscr{G}(P)$ alternates between the two parts of the bipartition. Therefore the difference between the number of vertices visited in one part and the other can not be greater than 1. The proof in case of a cycle goes analogously. □

Corollary

If $\mathscr{D}(P) > 1$ then $\mathscr{G}(P)$ has no Hamiltonian path. If $\mathscr{D}(P) > 0$ then $\mathscr{G}(P)$ has no Hamiltonian cycle.

Composition

Definition

For two disjoint posets P and Q we define $P \mid Q$ to be their parallel composition and PQ to be their series composition. We let i^n denote a chain of n elements labeled with i. In some cases the labeling of the poset is of no importance and in these cases we will write $\circ$.

This lets us write the set cases as posets of the form $\circ \mid \circ \mid \cdots$, and the multiset cases as posets of the form $\circ^{k_1} \mid \circ^{k_2} \mid \cdots$. Furthermore, posets of the form $\circ^{k_1} \mid \circ^{k_2}$ are called binary.

Forests

Definition

A forest is a poset such that for every element the subposet of all succeeding elements is a chain. Maximal elements are called roots and minimal elements leafs.

Forests generalize the multiset and set cases and can also be created with just parallel and series composition, more specifically using just $\cdot \mid \cdot$ and $\cdot\circ$, where the dots represents some forest to be filled in.

Example

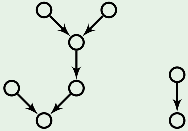


Figure: Hasse diagram of the forest $(\circ \mid (\circ \mid \circ) \circ \circ) \circ \mid \circ \circ$.

Properties of neighbour-swap graphs

Partitioning

Definition

For two poset P and Q, where Q is on a subset of the ground plane of P, define $P \leftharpoondown Q$ as their union.

Theorem

For any subposet $Q \subseteq P$ *we can partition* $\mathscr{L}(P)$ *into* $\{\mathscr{L}(P \leftharpoondown L) \mid L \in \mathscr{L}(Q)\}$

Proof.

Take an arbitrary linear extension of P. Within this linear extension the elements of Q have also been placed in an order which, obviously, corresponds with a linear extension of Q. □

Partitioning

Theorem

For any subposet $Q \subseteq P$ *we can partition* $\mathscr{G}(P)$ *into subgraphs* $\{\mathscr{G}(P \leftharpoondown L) \mid L \in \mathscr{L}(Q)\}$ *which are connected to each other isomorphic to* $\mathscr{G}(Q)$.

Corollary

Every neighbour-swap graph is isomorphic to a subgraph of a set case.

Partitioning

Example

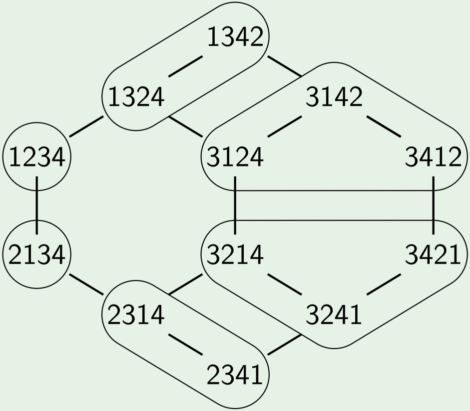


Figure: $\mathscr{G}(1 \mid 2 \mid 34)$ partitioned with $1 \mid 2 \mid 3$

Identification

Theorem

If a neighbour-swap graph contains a vertex of degree 1 then the graph is isomorphic to a subgraph of a binary case.

Proof.

A vertex of degree 1 corresponds with a linear extension in which only one neighbour-swap can be made. This means no neighbour-swap can be performed on the elements left of the swap i.e. they are ordered. Similarly, all the elements on the right of the swap are also ordered. We may conclude that the poset can be divided into two chains on which maybe some more orderings are placed. □

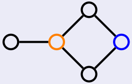
Conjecture

Every neighbour-swap graph has at at most two vertices of degree 1.

Identification

Theorem

If a neighbour-swap graph contains a vertex, highlighted in orange, of degree 3, then the graph has at least another vertex, highlighted in blue:



Proof.

A vertex of degree 3 corresponds with a linear extension in which three neighbour-swaps can be made. In the worst case these neighbour-swaps all “interfere” which each other, i.e. their locations are all right next to each other. This means the linear extension is of the form ...*abcd*... where *a*&*b*, *b*&*c* and *c*&*d* may be swapped. Aside from the obvious three linear extensions *bacd*, *acbd*, *abdc* we can be certain another exists, namely: *badc*. □

Identification

Theorem

If a neighbour-swap graph contains the following part:

i.e. a "tail", then the graph is trivial i.e. it is a path.

Edges & Trivia

Theorem

The number of edges in $\mathscr{G}(M)$, where $M = \circ^{k_1} \mid \circ^{k_2} \mid \cdots$ is a multiset case, equals:

$$\frac{n^2 - n_2}{2} \binom{n-1}{k_1, k_2, \cdots}$$

where $n = k_1 + k_2 + \cdots, n_2 = k_1^2 + k_2^2 + \cdots$.

Theorem

$\mathscr{G}\left(\left(\circ^Q \mid P\right) Q\right)$ is the more connected version of $\mathscr{G}(Q \mid P)$.

Counting linear extensions and their surplus

Counting linear extensions of compositions

Theorem

$$\left|\mathscr{L}\left(P \mid Q\right)\right| = \left|\mathscr{L}\left(P \mid \circ^{Q}\right)\right| \cdot \left|\mathscr{L}\left(Q\right)\right|$$

Theorem

$$\left|\mathscr{L}\left(PQ\right)\right| = \left|\mathscr{L}\left(P\right)\right| \cdot \left|\mathscr{L}\left(Q\right)\right|$$

Theorem

The number of linear extensions of a binary case is:

$$\left|\mathscr{L}\left(\circ^{k_1} \mid \circ^{k_2}\right)\right| = \binom{n}{k_1}$$

Example

To illustrate the use of the previous three theorems let us quickly count the number of linear extension of the poset $\circ\left(\circ \mid \circ \mid \circ\right)\left(\circ \mid \circ^3\right)$.

$$\begin{aligned}
\left|\mathscr{L}\left(\circ\left(\circ \mid \circ \mid \circ\right)\left(\circ \mid \circ^3\right)\right)\right| &= \left|\mathscr{L}\left(\circ\right)\right| \cdot \left|\mathscr{L}\left(\circ \mid \circ \mid \circ\right)\right| \cdot \left|\mathscr{L}\left(\circ \mid \circ^3\right)\right| \\
&= 1 \cdot \left|\mathscr{L}\left(\circ \mid \circ^2\right)\right| \cdot \left|\mathscr{L}\left(\circ \mid \circ\right)\right| \cdot 4 \\
&= 1 \cdot 3 \cdot 2 \cdot 4 \\
&= 24
\end{aligned}$$

Counting linear extensions of a forest

Theorem

The number of linear extensions of a forest F is:

$$|\mathscr{L}(F)| = \frac{n!}{d_1 d_2 \cdots}$$

where d_i is the number of preceding elements, including itself, of the i'th element.

Corollary

The number of linear extensions of a multiset case is:

$$\left|\mathscr{L}\left(\circ^{k_1} \mid \circ^{k_2} \mid \cdots\right)\right| = \binom{n}{k_1, k_2, \cdots}$$

The number of linear extensions of a set case is:

$$|\mathscr{L}(\circ \mid \circ \mid \cdots)| = n!$$

Surplus of composition

Theorem

$$\mathscr{D}(P \mid Q) = \mathscr{D}\left(P \mid \circ^{Q}\right) \cdot \mathscr{D}(Q)$$

Theorem

$$\mathscr{D}(PQ) = \mathscr{D}(P) \cdot \mathscr{D}(Q)$$

Example

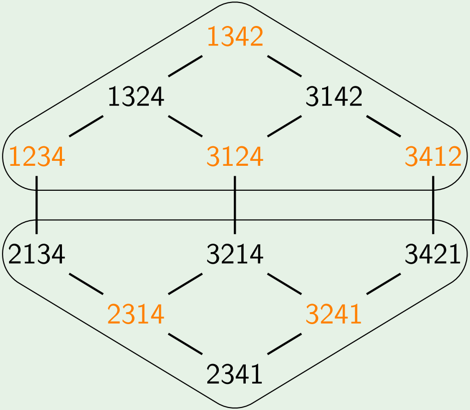


Figure: 1 | 2 | 34 partitioned with 1 | 2

Surplus of two odd chains

Theorem

$\mathscr{D}(\circ^n \mid \circ^m) = 0$ *for odd* n, m.

Proof.

For clarity we will abuse notation for a second and omit the chain and parallel notation.

$$\begin{aligned}\mathscr{D}(1, n-1, 1, m-1) &= \mathscr{D}(n, m) \cdot \mathscr{D}(1, n-1) \cdot \mathscr{D}(1, m-1) \\ &= \mathscr{D}(1, 1) \cdot \mathscr{D}(2, n-1, m-1) = 0\end{aligned}$$

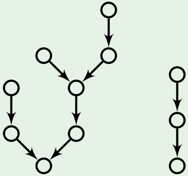
□

Surplus of a forest

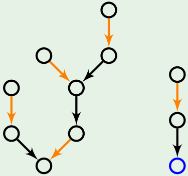
Definition

A pairing in a forest F is a(n) (almost) perfect matching of its transitive reduced Hasse diagram, leaving at most one root unmatched. A collapsed forest $\bar{F}$ is obtained by contracting the elements which are paired and deleting the unmatched root if necessary.

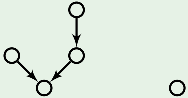
Example



(a) Forest



(b) Paired



(c) Collapsed

Surplus

Theorem

$$\mathscr{D}(F) = \begin{cases} \left|\mathscr{L}\left(\bar{F}\right)\right| & \text{if } F \text{ has a pairing} \\ 0 & \text{otherwise} \end{cases}$$

Corollary

$$\mathscr{D}\left(\circ^{k_1} \mid \circ^{k_2} \mid \cdots\right) = \begin{cases} \binom{n \div 2}{k_1 \div 2, k_2 \div 2, \cdots} & \text{if at most one } k \text{ is odd} \\ 0 & \text{if at least two } k \text{ are odd} \end{cases}$$

Constructing shortest covering walks

Steinhaus-Johnson-Trotter algorithm

Theorem

If $\mathscr{G}(P)$ has a Hamiltonian path then so does $\mathscr{G}(P \mid \circ)$.

Corollary

$\mathscr{G}(\circ \mid \circ \mid \cdots)$*, i.e. a set case, has a Hamiltonian path.*

Generalized Steinhaus-Johnson-Trotter algorithm

Theorem

If $\mathscr{G}(P)$ and $\mathscr{G}\left(\circ^P \mid \circ^k\right)$ have Hamiltonian paths then so does $\mathscr{G}\left(P \mid \circ^k\right)$.

Proof.

We partition $\mathscr{G}\left(P \mid \circ^k\right)$ into $\{\mathscr{G}\left(L_1 \mid \circ^k\right), \mathscr{G}\left(L_2 \mid \circ^k\right), \ldots\}$, where $\{L_1, L_2, \ldots\}$ is a Hamiltonian path of P, which are all isomorphic to $\mathscr{G}\left(\circ^P \mid \circ^k\right)$ which have, by assumption, Hamiltonian paths. The Hamiltonian paths in these subgraphs must have their ends at $L_j \circ^k$ and $\circ^k L_j$ because these vertices have degree 1. By connecting these endpoints the right way we create a Hamiltonian path through $\mathscr{G}\left(P \mid \circ^k\right)$. $\square$

Stachowiak

Theorem

If $\mathscr{G}(P)$ has a Hamiltonian path and $\mathscr{D}(P) = 0$ then so does $\mathscr{G}(P \mid Q)$ for every poset Q.

Assuming the neighbour-swap graph of two odd chains has a Hamiltonian path Stochawiak is able to easily prove the following:

Theorem

Let M be a multiset case. If $\mathscr{D}(M) \leq 1$ then $\mathscr{G}(M)$ has a Hamiltonian path.

Proof.

It can be deduced that $\mathscr{D}(M) = 1$ only happens when $\mathscr{G}(M)$ is trivial i.e. it is a path. When $\mathscr{D}(M) = 0$ there must exist at least two chains of odd length, which can then be used as the basis for Stachowiaks theorem to show $\mathscr{G}(M)$ has a Hamiltonian path. These two chains can be used as the basis because the surplus of two odd chains is indeed 0. □

Conjecture

Ruskey's Conjecture

If $\mathscr{D}(P) = 0$ then $\mathscr{G}(P)$ has a Hamiltonian path for every poset P.

Lehmer & Master

Definition

An unvisited vertex u at distance 1 can be visited by side-stepping from a vertex v on the path to u and then immediately back again to v. In the resulting path, v occurs twice, with only u in-between. Lehmer calls such a sidestep to an unvisited vertex at distance 1 a spur. A Lehmer path (cycle) in a graph is a path (cycle), possibly with single spurs, that visits the spur bases twice and all other vertices once.

Lehmer's Conjecture

$\mathscr{G}(M)$ has a Lehmer path with at most $\mathscr{D}(M) - 1$ single spurs for every multiset case M.

Master Conjecture

$\mathscr{G}(P)$ has a Lehmer path with at most $\mathscr{D}(P) - 1$ single spurs for every poset P.

Conclusion