Analysis and Approximation of Riemannian Distances on the 5D Homogeneous Space of 3D Positions and Orientations

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Chapter 1

Introduction

1.1 Synopsis research context

In many geometric image analysis applications one has to include multi-orientation image processing that relies on left-invariant Riemannian metrics and their logarithmic norm approximations.

For example, in diffusion-weighted MRI one obtains after some inverse methods processing an orientation density function of water molecules that are generally believed to follow the biological fibres in the brain. These inverse methods are effective, but involve false peaks that are not aligned with peaks in neighbouring spherical distributions. Therefore contextual image processing group convolution with heat-kernel approximations on the 5D homogeneous space $\mathbb{M}_3 = \mathrm{SE}(3)/\mathrm{SO}(2)$ of 3D positions and orientations is needed to clean the fibre structures in the data [1, 2], valuable in clinical applications [1, 3].

Furthermore, in many roto-translation equivariant deep learning methods one includes similarity metrics between points on $\mathbb{M}_3$ ("local orientations") where one again relies on left-invariant Riemannian metrics on the Lie group quotient $\mathbb{M}_3$ and their logarithmic norm approximations. For instance in [4] where "fast, expressive SE(3) equivariant neural networks" (G-CNNs) are achieved with many applications. There the neural network kernels follow symmetry considerations in geometric image processing [5].

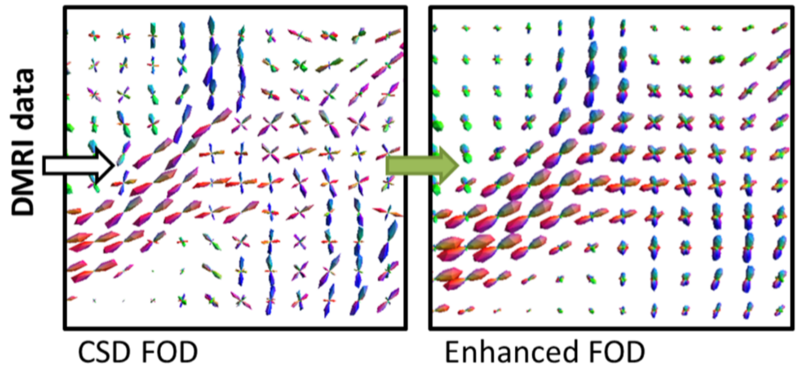
In these works and in [6, 7], a convenient choice of section in the quotient G/H is helpful for carrying differential geometrical tools from the group G towards G/H in a computationally tangible way. For $G = \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ and $H = \mathrm{SO}(2)$ we study a specific section $\sigma_{\mathrm{MAV}} : G/H \to G$ which will be explained later. We will also improve this specific section when needed by defining the better approximation σ_{NEW}.

1.2 Research context

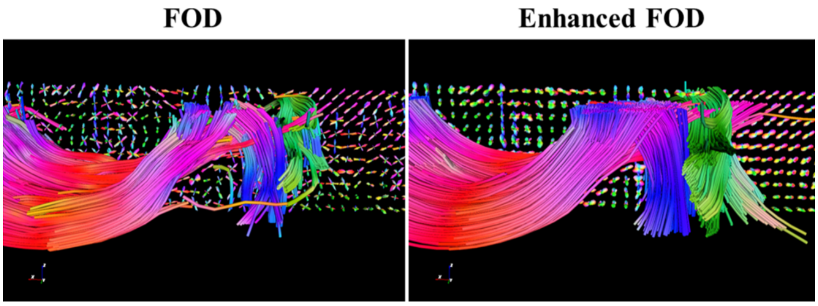
Denoising on position-orientation space

In Diffusion-Weighted Magnetic Resonance Imaging (DW-MRI) one obtains (for details see [8]) a probability density function $U : \mathbb{M}_3 \to \mathbb{R}$ on position-orientation space $\mathbb{M}_3 := \mathbb{R}^3 \times S^2$, where $U(\boldsymbol{p})$, with $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{n}) \in \mathbb{M}_3$, can be interpreted as the probability density that a water molecule is present at location $\boldsymbol{x}$ and moving in the direction of the orientation $\boldsymbol{n}$.

It is generally believed that water molecules follow the biological fibres in brain white matter. This means that from the map U it should be possible to reconstruct the biological fibres and as such the structural connectivity of the brain in general. Due to the nature of DW-MRI, its images are subject to noise [8], and denoising this data, see Fig. 1.1a and 1.1b, greatly improves fibre tracking.



(a) Denoising of a DW-MRI image. The left image is the noisy data U, and the right image is the denoised image. The data $U(\boldsymbol{p})$ are visualised by plotting the set $\{\boldsymbol{x} + \nu\, U(\boldsymbol{x}, \boldsymbol{n})\, \boldsymbol{n} \mid \boldsymbol{n} \in S^2\}$ for discrete points $\boldsymbol{x} \in \mathbb{Z}^3$, where we only view a 2D slice. The constant $\nu > 0$ is chosen such that neighbouring distributions do not overlap.



(b) Comparison of fibre tracking on DW-MRI data before and after denoising using group convolutions.

Figure 1.1: Comparison of DW-MRI data in (a) and fibre tracking on DW-MRI data in (b) before (left) and after (right) denoising with a Gaussian kernel. Images taken from Figures 6.1 and 6.7 of [8] respectively.

To denoise the data U a convolution with a Gaussian kernel $K_t : \mathbb{M}_3 \times \mathbb{M}_3 \to \mathbb{R}$ on $\mathbb{M}_3$ is performed:

$$W(\boldsymbol{p}, t) = \int_{\mathbb{M}_3} K_t(\boldsymbol{p}, \boldsymbol{q})\, U(\boldsymbol{q})\, \mathrm{d}\boldsymbol{q}, \qquad \boldsymbol{p}, \boldsymbol{q} \in \mathbb{M}_3,\ t > 0, \tag{1.1}$$

where the kernel K_t is such that the denoised data W is the solution of the diffusion PDE $\frac{\partial W}{\partial t} = \Delta_{\mathcal{G}} W$, with initial condition $\lim_{t \downarrow 0} W(\boldsymbol{p}, t) = U(\boldsymbol{p})$, and $t > 0$ is a parameter determining the amount of smoothing. Here $\Delta_{\mathcal{G}} = \mathrm{div}_{\mathcal{G}} \circ \mathrm{grad}_{\mathcal{G}}$ is the Laplace-Beltrami operator which depends on the Riemannian metric $\mathcal{G}$ on $\mathbb{M}_3$ (see also chapter 2 of [9]). In Figure 1.1a the green arrow represents the mapping $U \mapsto W(\cdot, t)$ and as we can see in Figure 1.1b this smoothing leads to an anatomically more plausible subsequent tractography.

Convolutions on position-orientation space

In general, a convolution on $\mathbb{M}_3$ is any linear operator Φ acting on functions $f : \mathbb{M}_3 \to \mathbb{R}$ as

$$(\Phi f)(\boldsymbol{p}) = \int_{\mathbb{M}_3} k(\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{q})\, \mathrm{d}\boldsymbol{q}, \qquad \boldsymbol{p}, \boldsymbol{q} \in \mathbb{M}_3, \tag{1.2}$$

where $k : \mathbb{M}_3 \times \mathbb{M}_3 \to \mathbb{R}$ is a kernel that is invariant under SE(3), i.e.

$$k(g \triangleright \boldsymbol{p}, g \triangleright \boldsymbol{q}) = k(\boldsymbol{p}, \boldsymbol{q}), \quad \forall g \in \mathrm{SE}(3), \tag{1.3}$$

where the *Special Euclidean group* SE(3) is the Lie group of rotations and translations, together referred to as rototranslations, on three-dimensional space $\mathbb{R}^3$, and $\triangleright : \mathrm{SE}(3) \times \mathbb{M}_3 \to \mathbb{M}_3$ is the natural action of SE(3) on $\mathbb{M}_3$, which will be given in Section 3.2. We write $G = \mathrm{SE}(3)$.

It is natural to process (via a convolution Φ) signals $f : \mathbb{M}_3 \to \mathbb{R}$ on $\mathbb{M}_3$ in a way that respects rototranslational symmetries. With this we mean that it should not matter whether the data is rototranslated before or after processing, the result should be the same. This is known as equivariance under the group SE(3), or SE(3)-equivariance. Mathematically, this means

$$\Phi(g \triangleright f) = g \triangleright (\Phi f). \tag{1.4}$$

Note that $\triangleright$ is also used to denote the action of the Lie group on functions defined on $\mathbb{M}_3$.

Importantly, due to the invariance of the kernel k, convolutional operators as in Eq. (1.2) on $\mathbb{M}_3$ are equivariant[5]. Specifically, denoising of the DW-MRI data U can be made to be equivariant. Namely, if the Riemannian metric $\mathcal{G}$ on $\mathbb{M}_3$ is constructed to be invariant, then the Gaussian kernel K_t is invariant, meaning that the denoising happens equivariantly.

Other applications of convolutions

Aside from denoising DW-MRI data, such convolutions on $\mathbb{M}_3$ are also used in SE(3) group equivariant convolutional neural networks (G-CNNs) such as the PONITA architecture [4] and the PDE-based CNN architecture [10, 11]. By using convolutions, visualised by the green arrows in Figure 1.2, on $\mathbb{M}_3$ the architectures are made to be equivariant. Equivariant models generally need less training data to achieve a desired accuracy, as rototranslational symmetries are already built in and do not need to be trained. This can be beneficial in situations where data is scarce.

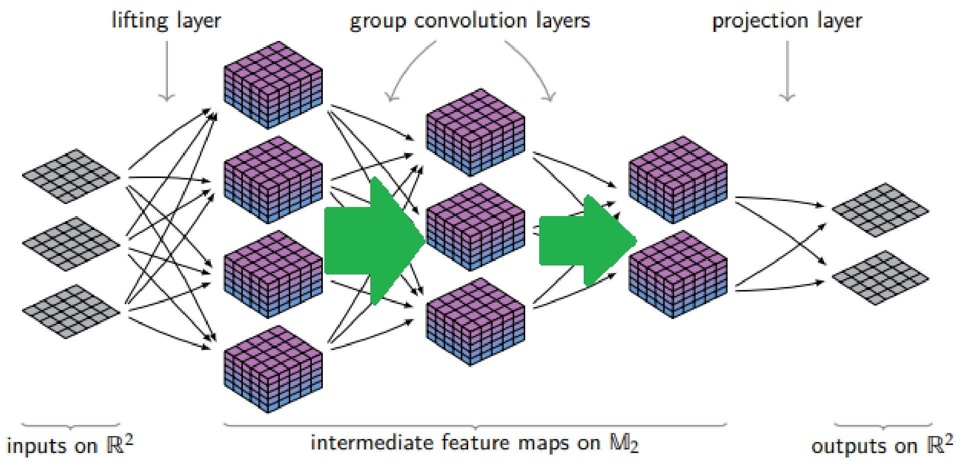


Figure 1.2: Schematic overview of a convolutional neural network. For simplicity the dimension has been reduced to 2. Image taken from Fig. 5.10 of [12].

Distance and approximations

In order to perform the denoising of data on $\mathbb{M}_3$ as expressed in Equation (1.1), we need access to the Gaussian kernel K_t. The exact Gaussian kernel is somewhat involved (see [13]), but a good approximation [8, Fig. 4.4] is

$$K_t(\boldsymbol{p},\boldsymbol{q}) \approx \hat{K}_t(\boldsymbol{p},\boldsymbol{q}) := N_t\, e^{-\frac{d_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q})^2}{4t}}, \quad \boldsymbol{p},\boldsymbol{q} \in \mathbb{M}_3, \tag{1.5}$$

where N_t is a normalisation constant such that $\int_{\mathbb{M}_3} \hat{K}_t(\boldsymbol{p},\boldsymbol{q})\,\mathrm{d}\boldsymbol{q} = 1$, and $d_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q})$ is the Riemannian distance between the points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{M}_3$ induced by the Riemannian metric $\mathcal{G}$ on $\mathbb{M}_3$.

The Riemannian distance $d_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q})$ between the points $\boldsymbol{p},\boldsymbol{q} \in \mathbb{M}_3$ is defined as the length $L_{\mathcal{G}}$ of the shortest curve between said points. That is:

$$d_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q}) = \inf_{\gamma \in \Gamma(\boldsymbol{p},\boldsymbol{q})} L_{\mathcal{G}}(\gamma), \quad \text{with} \quad L_{\mathcal{G}}(\gamma) := \int_0^1 \|\dot{\gamma}(t)\|_{\mathcal{G}}\,\mathrm{d}t, \tag{1.6}$$

where $\Gamma(\boldsymbol{p},\boldsymbol{q})$ is the set of all piecewise differentiable curves connecting $\boldsymbol{p}$ and $\boldsymbol{q}$ and where $\|\dot{\gamma}(t)\|_{\mathcal{G}} = \sqrt{\mathcal{G}_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t))}$. Both the Riemannian distance and curve length are induced by the Riemannian metric $\mathcal{G}$ that is chosen on $\mathbb{M}_3$.

Importantly, if the chosen Riemannian metric $\mathcal{G}$ is SE(3)-invariant, the induced Riemannian distance $d_{\mathcal{G}}$ is invariant as well, such that denoising, Eq. (1.1), with the approximative kernel $\hat{K}_t$ is still an equivariant operation: the kernel is invariant as it builds directly upon the invariant distance $d_{\mathcal{G}}$.

Unfortunately, this Riemannian distance is computationally very expensive [14, 15]. For this reason, approximations of the Riemannian distance d are used to perform the denoising of DW-MRI data in practice [13].

One such approximation is the logarithmic distance $\rho_{\mathcal{G}}$. It is defined as the length of the shortest (projected) exponential curve between the two given points:

$$\rho_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q}) = \inf_{\eta \in E(\boldsymbol{p},\boldsymbol{q})} L_{\mathcal{G}}(\eta), \tag{1.7}$$

where $E(\boldsymbol{p},\boldsymbol{q}) \subsetneq \Gamma(\boldsymbol{p},\boldsymbol{q})$ is the set of all *exponential* curves $[0,1] \ni t \mapsto \exp(tX) \triangleright \boldsymbol{p}$ connecting $\boldsymbol{p}$ and $\boldsymbol{q}$, i.e. $\exp(X) \triangleright \boldsymbol{p} = \boldsymbol{q}$. Here we have that $X \in \mathfrak{g} = \mathfrak{se}(3)$ is an element of the Lie algebra $\mathfrak{g} = T_e(G)$ and referred to as a generator of G. These generators generate group elements using the exponential map $\exp : \mathfrak{g} \to G$, i.e. as $g = \exp(X)$. The inverse of this exponential map is given by the (not necessarily globally unique) logarithmic map log. Further details will follow in Section 2.3.

As such $E(\boldsymbol{p},\boldsymbol{q})$ is a proper subset of $\Gamma(\boldsymbol{p},\boldsymbol{q})$, and it directly follows that $d_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q}) \leqslant \rho_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q})$. The difference between the shortest overall curve, or geodesic (for the distance $d_{\mathcal{G}}$) and the shortest exponential curve (for the logarithmic distance $\rho_{\mathcal{G}}$) is visible in Figure 1.3, which also shows that as the distance decreases, the logarithmic distance becomes a better approximation of the Riemannian distance.

The reason for choosing this approximation is that the lengths of the projected exponential curves η are easy to calculate. Namely, we have that (by Lemma 2.1)

$$L_{\mathcal{G}}(t \mapsto \exp(tX) \triangleright \boldsymbol{p}) = \|X \triangleright \boldsymbol{p}\|_{\mathcal{G}}, \tag{1.8}$$

where we again use the symbol $\triangleright$ for the infinitesimal action $\triangleright : \mathfrak{se}(3) \times \mathbb{M}_3 \to T\mathbb{M}_3$ of the Lie algebra on the manifold, defined by $X \triangleright \boldsymbol{p} := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\gamma(t) \triangleright \boldsymbol{p})$, where $\gamma : [0,1] \to \mathrm{SE}(3)$ is any smooth curve with tangent vector $\dot{\gamma}(0) = X$. This means Equation (1.7) can be written as

$$\rho_{\mathcal{G}}(\boldsymbol{p},\boldsymbol{q}) = \inf_{\exp(X) \triangleright \boldsymbol{p} = \boldsymbol{q}} \|X \triangleright \boldsymbol{p}\|_{\mathcal{G}}. \tag{1.9}$$

The aim of this thesis is to find (a decent approximation of) the generator X that is selected in the minimisation problem above.

This is done by a simple parametrisation which turns this curve optimisation problem into a single-parameter optimisation problem, as illustrated in Figure 1.4. This parametrisation is explained next.

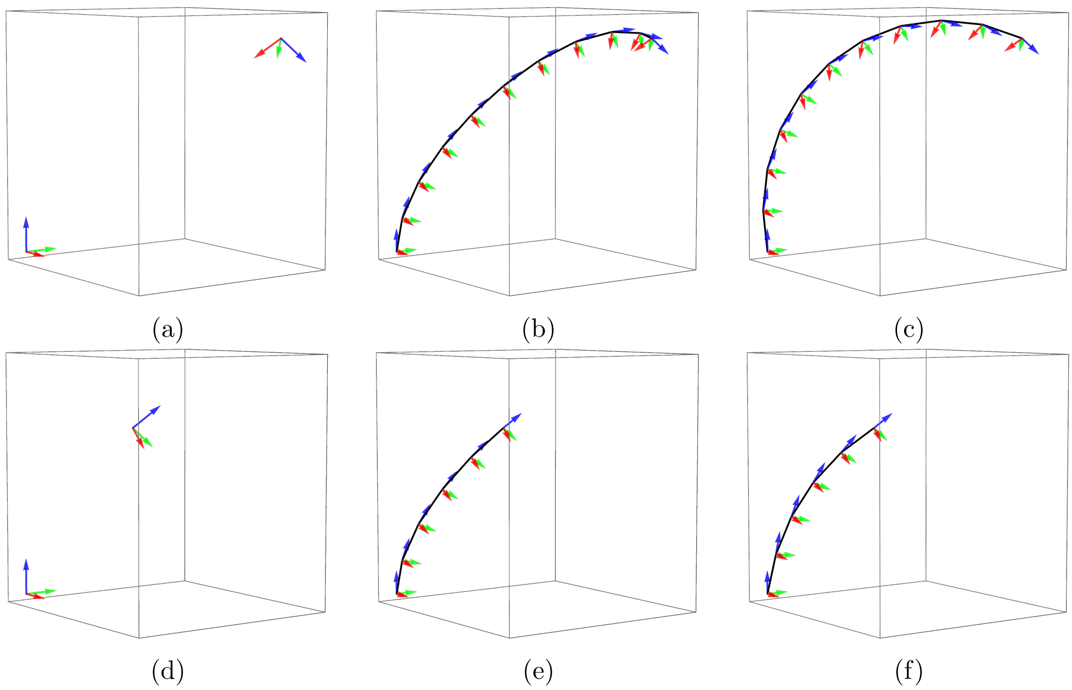
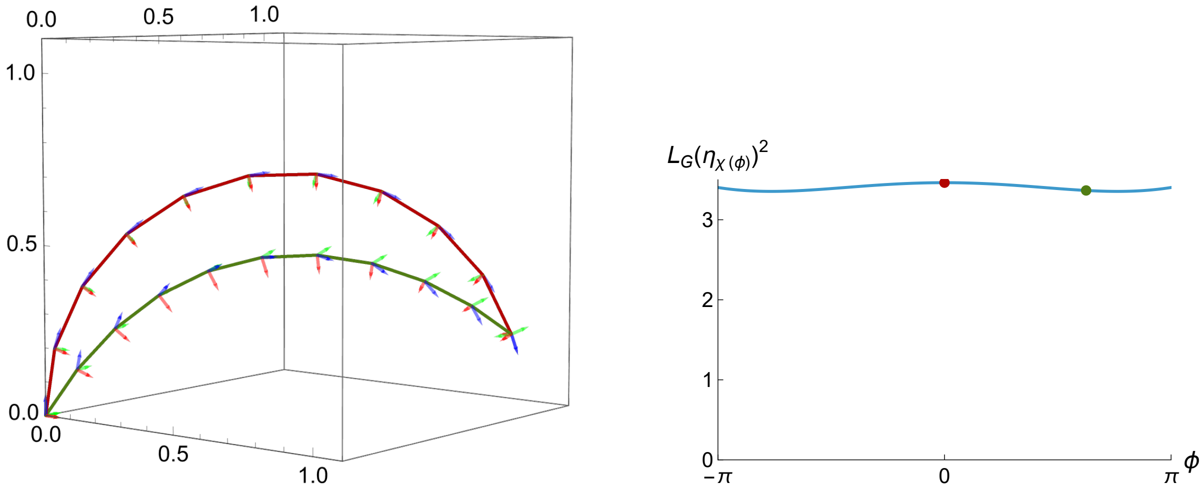


Figure 1.3: Left to right: (a) and (d): start-frame $\boldsymbol{p}_0$ (bottom left) and end-frames $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, (b) and (e): the exact geodesic connecting start- and end-frame, i.e. the curve minimising Equation (1.6), (c) and (f): the minimal exponential curve connecting start- and end-frame, i.e. the curve minimising Equation (1.7). All six subfigures plot the same area in space, such that all have the same scale. With this, notice that in the bottom row, the end-frame is closer (spatially) to the start-frame and the exponential curve in (f) is closer to the geodesic in (e) than in the row above. Here we visualise points $\boldsymbol{p} = (\boldsymbol{x}, \boldsymbol{n}) \in \mathbb{M}_3$ with a coordinate frame (of three orthogonal axes) at position $\boldsymbol{x} \in \mathbb{R}^3$, with the blue axis corresponding to orientation $\boldsymbol{n}$. Figures (a)-(c) taken from Figure 1 of [16] and (d)-(f) self-generated.



(a) All (only two shown) exponential curves $\eta_{\chi(\varphi)} : t \mapsto \exp(t\,\chi(\varphi))$ have the same boundary conditions $\boldsymbol{p}_0, \boldsymbol{p}_1 \in \mathbb{M}_3$ (start- and end-frames).

(b) Each exponential curve in (a) is parametrised by the parameter $\varphi \in [-\pi, \pi)$, with squared length $L^2_{\mathcal{G}}(\eta_{\chi(\varphi)})$ on the y-axis.

Figure 1.4: The optimisation problem (Eq. (1.7)) that is associated with computing the logarithmic distance $\rho_{\mathcal{G}}$ is a problem of curves. By indexing each exponential curve by a single parameter φ, this changes into a one-dimensional optimisation problem, as each curve in (a) corresponds to a single point (with same color) on the graph in (b). Figure adapted from [16].

Screw transformations

In comparison to other literature [1, 4, 5, 7, 8, 13, 16] where this problem is expressed in logarithmic coordinates, in this thesis we take a alternative perspective and describe rototranslations as screw transformations (see Figure 1.5 as well as [17]). Typically, a rototranslation is described as a mapping $\boldsymbol{x} \mapsto \boldsymbol{t} + \boldsymbol{R}\boldsymbol{x}$, where the rotation $\boldsymbol{R}$ is restricted to be about the origin of $\mathbb{R}^3$. Another way to write a rototranslation is as a mapping $\boldsymbol{x} \mapsto \boldsymbol{t}+\boldsymbol{c}+\boldsymbol{R}(\boldsymbol{x}-\boldsymbol{c})$ where now the rotation is about (an axis through) an arbitrary point $\boldsymbol{c} \in \mathbb{R}^3$. A screw transformation is a rototranslation written in this form

$$\boldsymbol{x} \mapsto \boldsymbol{t} + \boldsymbol{c} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{c}) \tag{1.10}$$

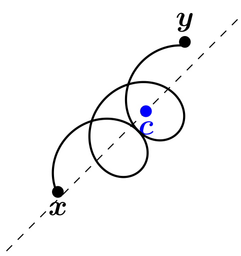


Figure 1.5: Screw motion from $\boldsymbol{x}$ to $\boldsymbol{y}$

but specifically with $\boldsymbol{R}\boldsymbol{t} = \boldsymbol{t}$. Hence, we image a screw transformation as rotating around and translating along the same axis $\boldsymbol{c} + \alpha\boldsymbol{t},\ \alpha \in \mathbb{R}$.

By Chasles' Theorem, every rototranslation can be written as a screw transformation. Importantly, acquiring a generator of rototranslation written as a screw transformation is easy, in contrast to finding a generator of a rototranslation written in another form. As we are interested in finding all generators that map between two position-orientations, using screw transformations is natural. Specifically, a generator of a screw transformation $\boldsymbol{x} \mapsto \boldsymbol{t} + \boldsymbol{c} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{c})$ is (written as a vector field) $\boldsymbol{x} \mapsto \boldsymbol{t}+\boldsymbol{M}_\omega(\boldsymbol{x}-\boldsymbol{c})$ where $\boldsymbol{M}_\omega \in \mathfrak{so}(3)$ is *any* generator of the rotation matrix $\boldsymbol{R} \in \mathrm{SO}(3)$[1], i.e. $\exp(\boldsymbol{M}_\omega) = \boldsymbol{R}$ in the matrix exponential, with axis of rotation $\boldsymbol{\omega} \in \mathbb{R}^3$. In other words, finding generators of a rototranslation reduces to solely finding generators in SO(3), which is substantially easier.

It will even lead to a better parametrisation of the fibres that occur when expressing

[1]The *Special Orthogonal group* SO(3) is the Lie group of 3D rotations with Lie algebra $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$.

$\mathbb{M}_3 \equiv \mathrm{SE}(3)/\mathrm{SO}(2)$ and a more comprehensive analysis of logarithmic norms. The key idea (generalising to other homogenous spaces) is invariant decomposition of the Lie-algebra $\mathfrak{se}(3)$ which enables one to write SE(3)-elements as a product of a translation and a rotation such that they commute.

Parametrisation of the screw transformation generators

As described earlier, we are interested in finding all generators X that map a position-orientation $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0)$ to $\exp(X) \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1) \in \mathbb{M}_3$. We do this by finding all screw transformations, Eq. (1.10), between the points $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$, and then finding their generators.

Due to the way the group action is defined, for the exact definition see Section 3.2, the first step is to find all rotations $\boldsymbol{R}$ that map

$$\boldsymbol{R}\boldsymbol{n}_0 = \boldsymbol{n}_1. \tag{1.11}$$

Every rotation $\boldsymbol{R}$ can be described as a combination of its angle and axis $(\theta, \boldsymbol{k})$. Specifically,

$$\boldsymbol{R} = \exp(\boldsymbol{M}_\omega) \quad \text{with angular velocity vector} \quad \boldsymbol{\omega} = \theta \boldsymbol{k}. \tag{1.12}$$

The correspondence between the matrix $\boldsymbol{M}_\omega$ and angular velocity vector $\boldsymbol{\omega}$ is that the matrix maps as $\boldsymbol{M}_\omega : \boldsymbol{x} \mapsto \boldsymbol{\omega} \times \boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^3$. All rotation axes $\boldsymbol{k} \in S^2$ that map $\boldsymbol{n}_0$ to $\boldsymbol{n}_1$ lie in the (intersection of $S^2 \subset \mathbb{R}^3$ with the) bisection plane between the two orientations. Hence, we can index these axes as $\boldsymbol{k}_\varphi$ with a single angle $\varphi \in [-\pi, \pi)$. We define the general axis $\boldsymbol{k}_\varphi$ as the linear combination

$$\boldsymbol{k}_\varphi = \cos\varphi \, \boldsymbol{k}_0 + \sin\varphi \, \boldsymbol{k}_{\frac{\pi}{2}}, \tag{1.13}$$

with orthogonal basis elements $\boldsymbol{k}_0$ and $\boldsymbol{k}_{\frac{\pi}{2}}$. These axes are defined as the axis perpendicular to, and the axis in the plane defined by $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ respectively, i.e.

$$\boldsymbol{k}_0 = \frac{\boldsymbol{n}_0 \times \boldsymbol{n}_1}{\|\boldsymbol{n}_0 \times \boldsymbol{n}_1\|}, \quad \text{and} \quad \boldsymbol{k}_{\pi/2} = \frac{\boldsymbol{n}_0 + \boldsymbol{n}_1}{\|\boldsymbol{n}_0 + \boldsymbol{n}_1\|}. \tag{1.14}$$

When this axis $\boldsymbol{k}_\varphi$ is fixed, the rotation angle θ_φ is also determined, as there is only one possible angle[2] θ_φ (modulo 2π) that maps $\boldsymbol{n}_0$ into $\boldsymbol{n}_1$ around this axis.

Because $\boldsymbol{R}\boldsymbol{t} = \boldsymbol{t}$, the translation vector $\boldsymbol{t}$ is in the same direction as the rotation axis, i.e. it is the projection of the total translation vector $\boldsymbol{x}_1 - \boldsymbol{x}_0$ onto the axis $\boldsymbol{k}_\varphi$, that is

$$\boldsymbol{t}(\varphi) = [(\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{k}_\varphi] \, \boldsymbol{k}_\varphi. \tag{1.15}$$

From Equation (1.10), we see that implicitly, the centre of rotation $\boldsymbol{c} = \boldsymbol{c}(\varphi)$ is given by

$$\boldsymbol{R}(\varphi)(\boldsymbol{x}_0 - \boldsymbol{c}(\varphi)) + \boldsymbol{t}(\varphi) = \boldsymbol{x}_1 - \boldsymbol{c}(\varphi). \tag{1.16}$$

The explicit derivation is given in Section 4.2.

All in all, all rototranslations between two position-orientations can be found in screw transformation form and parametrised using a single parameter φ. Finding generators of a screw transformation is straightforward, meaning that now we can parametrise all generators between two position-orientations. We collect these in the mapping $\chi_{\boldsymbol{p}_0, \boldsymbol{p}_1} : [-\pi, \pi) \to \mathfrak{se}(3)$, which, in summary, is given by the following Lemma.

[2] When keeping the direction of rotation consistent, as $2\pi - \theta_\varphi$ in the opposite direction also is a possible angle.

Lemma 1.1 (Parametrisation of the screw transformation generators)**.** *The family of generators $\chi = \chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi)$ that map a position-orientation $\mathbf{p}_0 = (\mathbf{x}_0, \mathbf{n}_0)$ to another position-orientation $\exp(\chi) \triangleright \mathbf{p}_0 = \mathbf{p}_1 = (\mathbf{x}_1, \mathbf{n}_1) \in \mathbb{M}_3$, can be parametrised as:*

$$\mathfrak{se}(3) \ni \chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi) : (\mathbf{x}_0, \mathbf{n}_0) \mapsto (\mathbf{t} + \mathbf{M}_{\omega(\varphi)}(\mathbf{x}_0 - \mathbf{c}), \mathbf{M}_{\omega(\varphi)}\mathbf{n}_0), \tag{1.17}$$

where the generated group element is given by

$$\mathrm{SE}(3) \ni \exp(\chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi)) : (\mathbf{x}_0, \mathbf{n}_0) \mapsto (\mathbf{t} + \mathbf{c} + \mathbf{R}(\mathbf{x}_0 - \mathbf{c}), \mathbf{R}\mathbf{n}_0) = \mathbf{p}_1. \tag{1.18}$$

Here we have that $\mathbf{R} = \mathbf{R}(\varphi)$ is given in Equation (1.11) with $\exp(\mathbf{M}_{\omega(\varphi)}) = \mathbf{R}$, and $\mathbf{t} = \mathbf{t}(\varphi)$ and $\mathbf{c} = \mathbf{c}(\varphi)$ are given by Equations (1.15) and (1.16) respectively.

The full derivation of these generators (and thus proof of this Lemma) will follow in Section 4.2.

Reparametrising the logarithmic distance

Through this novel parametrisation of all generators between two position-orientations $\mathbf{p}_0$ and $\mathbf{p}_1$, we can reparametrise the logarithmic distance approximation in Equation (1.9) as

$$\rho_{\mathcal{G}}(\mathbf{p}_0, \mathbf{p}_1) = \inf_{\varphi \in [-\pi,\pi)} \|\chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi) \triangleright \mathbf{p}_0\|_{\mathcal{G}}. \tag{1.19}$$

Keep in mind this is actually an optimisation of curves, see Figure 1.4. As we are interested in the minimising generator $\chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi)$, the objective function above demands further analysis. To avoid issues with differentiability, we will actually analyse its square, denoted by

$$\boxed{E_{\mathbf{p_0},\mathbf{p_1}}(\varphi) := \|\chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi) \triangleright \mathbf{p}_0\|^2_{\mathcal{G}}.} \tag{1.20}$$

We have put the points $\mathbf{p}_0, \mathbf{p}_1$ in subscript as we like to emphasise that these are kept constant throughout the analysis. When these points are clear, the subscript will be removed. The function $\sqrt{E(\varphi)} = \|\chi(\varphi) \triangleright \mathbf{p}_0\|_{\mathcal{G}}$ (for specific parameters $\mathbf{p}_0, \mathbf{p}_1$ and metric $\mathcal{G}$) is visible in Figure 1.6.

With our current knowledge, the minimisation over all generators, i.e. angles φ in Equation (1.19) is still computationally difficult. For this reason we restrict ourselves to a specific generator and investigate whether this is a (approximate) minimiser. Namely, we choose the generator with Minimal Angular Velocity (as we will show in Section 1.3), or the MAV generator, $M(\mathbf{p}_0, \mathbf{p}_1) \in \mathfrak{se}(3)$ [17] and define the MAV distance approximation as

$$d_{\mathrm{MAV}}(\mathbf{p}_0, \mathbf{p}_1) = \|M(\mathbf{p}_0, \mathbf{p}_1) \triangleright \mathbf{p}_0\|_{\mathcal{G}}. \tag{1.21}$$

It is clear that $\rho_{\mathcal{G}}(\mathbf{p}_0, \mathbf{p}_1) \leqslant d_{\mathrm{MAV}}(\mathbf{p}_0, \mathbf{p}_1)$ as we specialised our situation again, and the hope is that d_{MAV} is close to $\rho_{\mathcal{G}}$. Moreover, the parametrization $\chi_{\mathbf{p_0},\mathbf{p_1}}(\varphi)$ of the generators is such that the MAV generator corresponds to $\varphi = 0$, i.e. $\chi_{\mathbf{p_0},\mathbf{p_1}}(0) = M(\mathbf{p}_0, \mathbf{p}_1)$, meaning that $d^2_{\mathrm{MAV}} = E(0)$. With this, the problem of whether d_{MAV} is close to $\rho_{\mathcal{G}}$ has been reduced to the analysis of (the minima of) the function E. This is visible in Figure 1.6, where the values of these functions are given by the values of E at the labels σ_{MAV} and σ_ρ respectively.

Furthermore, we make two distinctions that may simplify the behaviour of $E(\varphi)$. The pair of position-orientations $\mathbf{p}_0$ and $\mathbf{p}_1$ are said to be coplanar if the translation vector $\mathbf{x}_1 - \mathbf{x}_0$ lies in the plane spanned by $\mathbf{n}_0$ and $\mathbf{n}_1$. Otherwise these are said to be non-coplanar. We also make the distinction between (spatially) isotropic and anisotropic metrics $\mathcal{G}$, based on the parameters defining the metric. If the spatial parameters are equal, the metric is spatially isotropic.

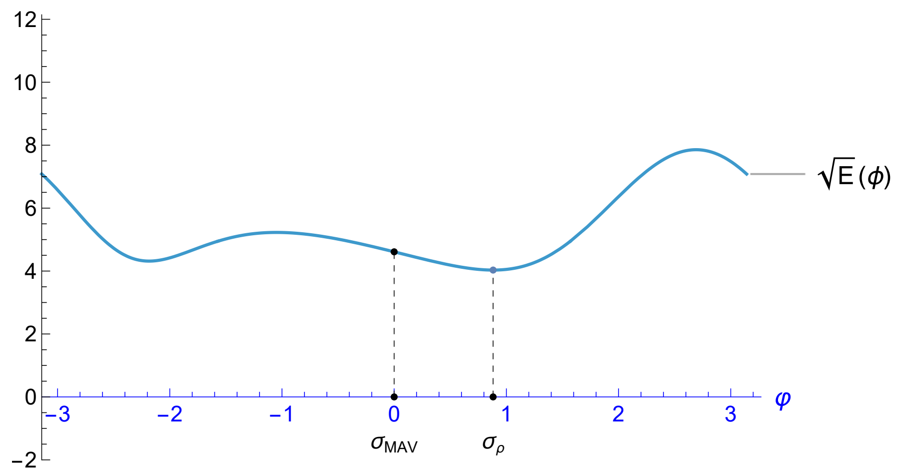


Figure 1.6: A possible plot of the function $\sqrt{E_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi)} = \|\chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}$ for specific position-orientations $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$, with (the values of φ corresponding to) the sections σ_{MAV} and σ_ρ, i.e. $\varphi = 0$ and the minimiser φ^* of $\sqrt{E}$. The φ-axis is blue to show its correspondence to a single fibre in G/H, schematically represented by the blue fibre in Figure 1.7.

1.3 Sections

In the previous discussion we only considered this problem on the manifold $\mathbb{M}_3$. The common approach, however, is to interpret the manifold as a Lie group quotient, such that the whole discussion can be done using group elements. This approach also allows the use of sections, which we introduce now.

When fixing a reference position-orientation $\boldsymbol{p}_0 = (\boldsymbol{0}, \boldsymbol{n}_0) \in \mathbb{M}_3$ one can view the space $\mathbb{M}_3$ as a group quotient, or homogeneous space G/H with $G = \mathrm{SE}(3)$ and subgroup $H = \mathrm{stab}_G(\boldsymbol{p}_0)$ of G, i.e. H consists of all rototranslations that leave $\boldsymbol{p}_0$ unchanged, i.e. 2D rotations around the axis $\boldsymbol{n}_0$: $H \equiv \mathrm{SO}(2)$. In this group quotient, every position-orientation $\boldsymbol{p} \in \mathbb{M}_3$ can be associated with the equivalence class, or coset, of rototranslations that map $\boldsymbol{p}_0$ to $\boldsymbol{p}$, i.e. $[g] = gH$ for any $g \in \mathrm{SE}(3)$ such that $g \triangleright \boldsymbol{p}_0 = \boldsymbol{p}$. This coset gH is also referred to as a fibre. In fact, by defining the projection map $\pi : G \to G/H$ as $g \mapsto [g] \cong g \triangleright \boldsymbol{p}_0$, the group G can be seen as a fibre bundle over G/H. This group quotient is schematically visualised in Figure 1.7.

Remark. *For every metric $\mathcal{G}$ on $M = G/H$ there exists a pseudometric $\widetilde{\mathcal{G}} = \pi^*\mathcal{G}$ on the group G such that $\widetilde{\mathcal{G}}$ is legal, i.e. it is invariant under left-multiplication by G and right-multiplication by H. This means that it is a metric when considered on G/H. Details will follow in Lemma 2.2.*

This projection map π can be used to define sections σ of G, which are maps $\sigma : G/H \to G$ with the condition that $\pi \circ \sigma = \mathrm{Id}_{G/H}$. In this case sections are maps that, when given any point $\boldsymbol{p} \in \mathbb{M}_3$, select one of the rototranslations in the fibre mapping from the reference $\boldsymbol{p}_0$ to $\boldsymbol{p}$. When the Lie group G has been given the corresponding pseudometric $\widetilde{\mathcal{G}}$, such a section may choose the rototranslation whose distance $d_{\widetilde{\mathcal{G}}}$ or logarithmic distance $\rho_{\widetilde{\mathcal{G}}}$ (defined similarly as in Equations (1.6) and (1.7)) to the identity element e is minimal, which we denote by σ_d and σ_ρ respectively:

$$\sigma_d([g]) = \operatorname*{arg\,min}_{h \in [g]} d_{\widetilde{\mathcal{G}}}(h, e), \quad \sigma_\rho([g]) = \operatorname*{arg\,min}_{h \in [g]} \rho_{\widetilde{\mathcal{G}}}(h, e). \tag{1.22}$$

Remark. *As shown in [16, Thm. 1, 2], in the above optimisations one may replace the pseudometric with any other legal (non-pseudo) metric, as for legal metrics geodesics have constant momentum and zero acceleration along the fibres.*

The section σ_{MAV} is the section that selects the fibre element with Minimal Angular Velocity:

$$\sigma_{\text{MAV}}([g]) = \underset{h \in [g]}{\arg\min} \, \|\boldsymbol{\omega}(h)\|, \tag{1.23}$$

$$= \underset{h \in [g]}{\arg\min} \, \rho_{\widetilde{\mathcal{G}}}(P_{\mathfrak{so}(3)} \log h, e) \tag{1.24}$$

where we define $\boldsymbol{\omega}(h)$ to be the angular velocity of the generator of h, see also Equation (1.12). Thus, the selected group element is generated by the MAV generator $M(\boldsymbol{p}_0, \boldsymbol{p})$:

$$\sigma_{\text{MAV}}([g]) = \exp\left(M(\boldsymbol{p}_0, \boldsymbol{p})\right), \tag{1.25}$$

for the fibre $[g]$ mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}$. It may also be defined by the condition in Euler angles

$$\alpha = -\gamma \Leftrightarrow \varphi = 0, \tag{1.26}$$

as these definitions coincide [16, Remark 2].

In Theorem 3 of the same paper [16], it has been shown that σ_{MAV} is locally close to σ_ρ, as illustrated in Figure 1.7, and in this thesis we further quantify this statement. We do this by studying when these sections coincide, i.e. when the MAV generator is a minimiser of $\rho_{\mathcal{G}}$. In the coplanar case we prove that the MAV generator always defines an extremum of $\rho_{\mathcal{G}}$, and we give the set on which this extremum is a minimum. In the non-coplanar case we define a better approximation of the minimiser, the section σ_{NEW}, obtained by minimising the third order Taylor series of E in φ, see Figure 1.8 for an example.

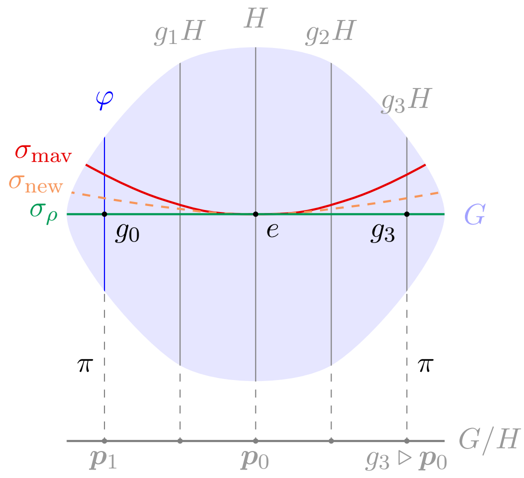


Figure 1.7: Schematic overview of the Lie group G with fibres $H = eH, g_1H, \cdots g_3H$ and the projection (π, dashed lines downwards) onto the group quotient G/H. The sections $\sigma_\rho, \sigma_{\text{NEW}}$ and σ_{MAV} select from each fibre a single element. Notice that for fibres close to the fibre eH the element selected by σ_{MAV} is close to that of σ_ρ, but for fibres further out the error grows. We propose the new section σ_{NEW} that is a better approximation of σ_ρ. Notice that all elements of G that map $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$ are indexed by the parameter φ and lie on the blue axis, corresponding to the blue axis in Fig. 1.6.

Notice that this homogeneous space requires the reference point $\boldsymbol{p}_0$ to remain fixed throughout. We will not use this, as instead we use the parametrisation of the previous section with auxiliary function $E(\varphi)$, since the problem has been rephrased to one of the minimisers of E. This way also allows any arbitrary initial position-orientation $\boldsymbol{p}_0$. For simplicity we do make the assumption $\boldsymbol{x}_0 = \boldsymbol{0}$.

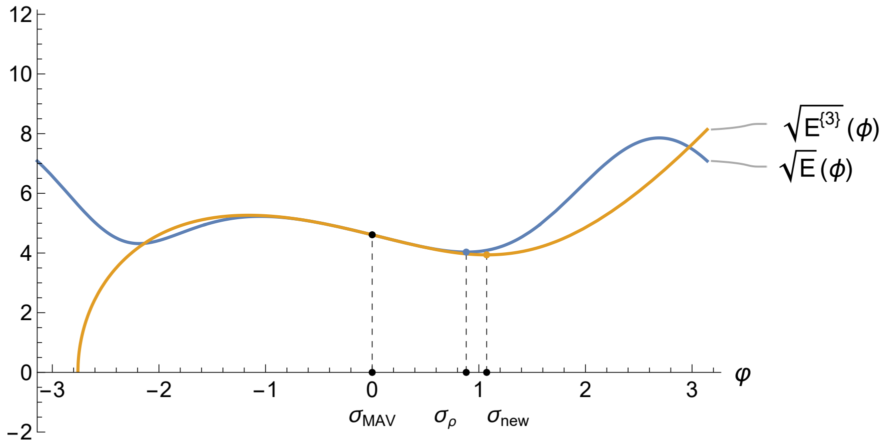


Figure 1.8: Example plot of the function $\sqrt{E_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi)} = \|\chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}$, for some specific choice of position-orientations $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$, together with its third order Taylor series, denoted by $\sqrt{E^{\{3\}}(\varphi)}$, in the orange line and (the values of φ corresponding to) the sections σ_{MAV} and σ_ρ, as well as the new section σ_{NEW}, the minimiser of the Taylor series.

1.4 The objective: minimising the logarithmic norm for screw motions in the fibre

In Lemma 2.3 and Corollary 4.2 we see that, apart from its initial definition as given in Equation (1.20), the function $E = E_{\boldsymbol{p}_0,\boldsymbol{p}_1}$ to optimise can be expressed as

$$\boxed{E(\varphi) = \|\chi(\varphi)\|^2_{\widetilde{\mathcal{G}}} = \|\log e^{\chi(\varphi)}\|^2_{\widetilde{\mathcal{G}}} = \rho_{\widetilde{\mathcal{G}}}(e^{\chi(\varphi)}, e)^2,} \tag{1.27}$$

where $\chi(\varphi) = \chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi)$. This means that $E(\varphi)$ is equal to the (squared) norm of the generator $\chi(\varphi)$ in the induced pseudometric $\widetilde{\mathcal{G}}$, i.e. $E(\varphi) = \widetilde{\mathcal{G}}_e(\chi(\varphi), \chi(\varphi)) = \|\chi(\varphi)\|^2_{\widetilde{\mathcal{G}}}$. This reveals the equivalence of taking the norm of the generator $\chi(\varphi)$ and taking the norm of the same generator acting on $\boldsymbol{p}_0$. Furthermore, the norm of generator $\chi(\varphi)$ is equal to the logarithmic distance (on G) between the generated element $\exp(\chi(\varphi))$ and the identity element e. Hence, since the fibre is parametrised by φ, we have that minimising $E(\varphi)$ minimises the logarithmic norm over the fibre that projects to $\boldsymbol{p}_1$. This correspondence is illustrated in the blue axis and fibre in Figures 1.6 and 1.7.

Notice that when minimising $E(\varphi)$ we minimise the full logarithmic norm over the fibre, whereas the MAV section only minimises the angular velocity over the fibre, see also Equation (1.23). In this light it is surprising that σ_{MAV} even approximates σ_ρ when close to the fibre $[e]$, as it only considers three out of the six dimensions.

1.5 Contributions

Generally speaking, we analyse the minima of the function E in both the coplanar and non-coplanar cases of points, and for both spatially isotropic and anisotropic metrics. In particular, we

- parametrise all generators between position-orientations in a novel way through $\chi(\varphi)$.
- analyse for which pairs of position-orientations $E(\varphi)$ is an even function (Lemma 4.2), in which case $\varphi = 0$, i.e. the MAV generator, is always an extremum (Thm. 4.1).

- compute first and second order derivatives of E at $\varphi = 0$ for both the spatially isotropic (Thm. 4.2) and spatially anisotropic metric (Thm. 4.3).
- use the second derivative of E to characterise the set on which E is minimal for $\varphi = 0$ in the coplanar case, for both spatially isotropic (Thm 5.1) and spatially anisotropic metrics (Thm. 5.2).
- also calculate the third order derivative of E at $\varphi = 0$ in order to provide (third-order) Taylor expansions of $E(\varphi)$ around $\varphi = 0$ in the non-coplanar case, for general metrics $\mathcal{G}$ (Lemma 5.5).
- use this Taylor expansion to find a better approximation of the minimiser φ_* of E (Thm. 5.3), as well as an upper bound of this approximation (Lma. 5.6), and use this to define the better (partial) approximation σ_{NEW} of σ_ρ, when compared to the old approximation σ_{MAV} (Thm. 5.4), and extend this to a full section $\overline{\sigma}_{\text{NEW}}$ (Thm. 5.5). We recommend using this improved approximation in applications.

Furthermore, we were also of help in proving a claim about legal metrics in [16]; expressed here as Lemma 2.4.

1.6 Structure

In Chapter 2 we introduce several preliminaries on differential geometry and Lie theory, as well as homogeneous spaces and fibres. With these we introduce the manifold $\mathbb{M}_3$ and Lie group SE(3) in Chapter 3. After this in Chapter 4 we introduce the norm in the usual coordinates and show the difficulties of computing this, and introduce the new parametrisation in which these computations simplify. In this chapter we also compute the first and second order derivative of $E(\varphi)$. In Chapter 5 we analyse the first and second derivative of $E(\varphi)$ to find when $\varphi = 0$ (the MAV generator) is a minimiser of E in the coplanar case. Later in this chapter we give the third-order Taylor series of E, and use this to find an approximation of the minimiser in the non-coplanar case. This thesis is concluded by Chapter 6, where we provide a summary as well as a possible generalisation.

Chapter 2

Preliminaries

In this chapter we will give a brief introduction to the subjects of Differential Geometry, Riemannian Geometry and Lie Theory, which will be needed to understand the manifold $\mathbb{M}_3$ and Lie group SE(3). These are introduced in a later chapter and will be used throughout the remainder of this report.

2.1 Differential Geometry

The following section will be used to give a brief overview of relevant ideas from the field of Differential Geometry. This section is based on [18], to which the reader is also referred for further explanation.

The most important object in the study of differential geometry is a (smooth) manifold, which, in simplified terms, is a space which locally looks like a Euclidean space. In this report we are only interested in real-valued manifolds, such that Euclidean space is given by $\mathbb{R}^n$ for some $n \in \mathbb{N}$. The manifold we discuss will always be smooth. The definition of a general, not necessarily smooth, manifold is given below.

Definition 2.1 (Smooth manifold)**.** A topological space M is a smooth manifold of dimension $n \in \mathbb{N}$ if

- M is a Hausdorff space, i.e. for every pair of distinct points $\boldsymbol{p}, \boldsymbol{q} \in M$, there exist disjoint open subsets $U, V \subseteq M$ such that $\boldsymbol{p} \in U, \boldsymbol{q} \in V$.
- There exists a countable basis for the topology of M.
- M is locally Euclidean of dimension n, i.e. for each $\boldsymbol{p} \in M$ and the open subset $U \subseteq M$ containing $\boldsymbol{p}$, there also exist an open subset $\widetilde{U} \in \mathbb{R}^n$ and a bijective function $\psi : U \to \widetilde{U}$, called a "chart", such that both ψ and ψ^{-1} are continuous.
- For any two charts $\psi_1 : U_1 \to \tilde{U}_1, \psi_2 : U_2 \to \tilde{U}_2$ that overlap on $U := U_1 \cap U_2 \neq \emptyset$, the "transition maps" $\psi_1 \circ \psi_2^{-1} : \psi_2(U) \to \psi_1(U)$ and $\psi_2 \circ \psi_1^{-1} : \psi_1(U) \to \psi_2(U)$ are smooth (in the sense of infinitely differentiable on $\mathbb{R}^n$).

Definition 2.2. A map $F : M_1 \to M_2$ between two smooth manifolds M_1, M_2, is called "smooth", if for any two charts ψ_1, ψ_2 on M_1, M_2, respectively, the mapping $\psi_2 \circ F \circ \psi_1^{-1}$ is smooth (in the sense of infinitely differentiable on $\mathbb{R}^n$).

Definition 2.3 (Diffeomorphism)**.** A diffeomorphism is a bijective map $F : M \to N$ between two smooth manifolds M and N, where both F and F^{-1} are smooth. If such a map exists, the manifolds are said to be diffeomorphic.

Remark. *For a smooth manifold M and smooth chart (U, ψ), the map $\psi : U \to \psi(U)$ is a diffeomorphism.*

Since manifolds are locally Euclidean, many manifolds can be "embedded" in a higher dimensional Euclidean space, i.e. visualised as a subset of $\mathbb{R}^m$ for some higher $m > n$. More specifically, an embedding is defined as

Definition 2.4. For smooth manifolds M and N, a smooth embedding of M into N is a smooth immersion[1] $F : M \to N$ that is also a homeomorphism onto its image.

Remark. *The most important example is the inclusion map $U \hookrightarrow M$ for $U \subseteq M$, where for visualisation we are often interested in $M = \mathbb{R}^n$.*

If the manifold M is embedded in some $\mathbb{R}^n$, a tangent vector $\boldsymbol{v}$ at some point $\boldsymbol{p} \in M$ can be visualised as the vector in $\mathbb{R}^n$ tangent to M at $\boldsymbol{p}$. The tangent space $T_{\boldsymbol{p}}M$ at $\boldsymbol{p}$ is the collection of all tangent vectors at $\boldsymbol{p}$, i.e. the (hyper)plane that is tangent to M at $\boldsymbol{p}$. We would, however, also like to define tangent vectors as an intrinsic property of the manifold, independent of its embedding. This is given in the following definition.

Definition 2.5 (Tangent vectors, tangent spaces and tangent bundles)**.** In general, tangent vectors are defined as the speeds of curves passing through $\boldsymbol{p}$. This means that for any curve $\gamma : [0, 1] \to M$ with $\gamma(0) = \boldsymbol{p}$, we have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) = \dot{\gamma}(0) \tag{2.1}$$

is a tangent vector at $\boldsymbol{p}$. These are unique up to an equivalence class, as there may be multiple curves passing through $\boldsymbol{p}$ with equal speed, and the quotient under this equivalence relation is the tangent space $T_{\boldsymbol{p}}M$ at $\boldsymbol{p}$ of M.

The tangent bundle TM of a manifold M is defined as the (disjoint) union of tangent spaces $T_{\boldsymbol{p}}M$ at all points $\boldsymbol{p} \in M$, usually written as $TM := \{(\boldsymbol{p}, \dot{\boldsymbol{p}}) \mid \boldsymbol{p} \in M, \dot{\boldsymbol{p}} \in T_{\boldsymbol{p}}M\}$.

Tangent vectors may also be used as partial derivatives, acting on functions f on M by

$$\dot{\gamma}(0)f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)). \tag{2.2}$$

Definition 2.6 (Differential)**.** For smooth manifolds M and N, and the smooth map $F : M \to N$ between the two, we can define at each point $\boldsymbol{p} \in M$ the differential of F as

$$\mathrm{d}F_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to T_{F(\boldsymbol{p})}N, \tag{2.3}$$

which for tangent vectors $\boldsymbol{v} \in T_{\boldsymbol{p}}M$ acts on functions f on N via $\mathrm{d}F_{\boldsymbol{p}}(\boldsymbol{v})(f) = \boldsymbol{v}(f \circ F)$.

Remark. *The differential at any specific point may also be called a "pushforward", and written as F_*.*

Definition 2.7 (Vector field)**.** A vector field X on manifold M is a continuous map

$$X : M \to TM, \qquad \boldsymbol{p} \mapsto X_{\boldsymbol{p}} \in T_{\boldsymbol{p}}M, \tag{2.4}$$

i.e. it associates a tangent vector to each element of the manifold.

Remark. *If the map $F : M \to N$ is a diffeomorphism, the pushforward of the vector field X on M by F is again a vector field.*

[1] Its differential (see Def. 2.6) is injective

2.2 Riemannian geometry

The next section will be used to give a brief overview of the field of Riemannian Geometry, where the smooth structure placed on the manifold is given by a Riemannian metric, defined below. This section is based on [9], in which also further explanation can be found.

Definition 2.8 (Riemannian metric)**.** A Riemannian metric $\mathcal{G}$ is a collection of inner products $\mathcal{G}_{\boldsymbol{p}}$ indexed by $\boldsymbol{p} \in M$ on the tangent spaces $T_{\boldsymbol{p}}M$. This means that for any element $\boldsymbol{p} \in M$ we have that $\mathcal{G}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \times T_{\boldsymbol{p}}M \to \mathbb{R}$ is a (real-valued) inner product on $T_{\boldsymbol{p}}M$, i.e. it is bilinear, symmetric and positive definite.

This inner product, like any, gives rise to a norm on $T_{\boldsymbol{p}}M$ via

$$\mathcal{G}_{\boldsymbol{p}}(\boldsymbol{v}, \boldsymbol{v}) = \|\boldsymbol{v}\|^2_{\mathcal{G}}. \tag{2.5}$$

Definition 2.9 (Riemannian manifold)**.** A Riemannian manifold is a pair $(M, \mathcal{G})$ of a manifold M and a Riemannian metric $\mathcal{G}$.

We would like to use Riemannian metrics to determine the length of any (piecewise) smooth curve $\gamma : [0,1] \to M$ on the Riemannian manifold. Recall from Definition 2.5 that for each $t \in [0,1]$, we have that $\dot{\gamma}(t) \in T_{\gamma(t)}M$.

Definition 2.10 (Length)**.** For a Riemannian manifold $(M, \mathcal{G})$, and a (piecewise) smooth curve $\gamma : [0,1] \to M$, the length of the curve γ is given by

$$L_{\mathcal{G}}(\boldsymbol{\gamma}) := \int_0^1 \|\dot{\boldsymbol{\gamma}}(t)\|_{\mathcal{G}} \mathrm{d}t = \int_0^1 \sqrt{\mathcal{G}_{\boldsymbol{\gamma}(t)}(\dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t))} \mathrm{d}t. \tag{2.6}$$

With this, we can define the distance between any pair of points $\boldsymbol{p}, \boldsymbol{q} \in M$ as the shortest curve length connecting these two points:

Definition 2.11 (Distance)**.** For any $\boldsymbol{p}, \boldsymbol{q} \in M$ the distance between $\boldsymbol{p}$ and $\boldsymbol{q}$ in the metric $\mathcal{G}$ is given by $d_{\mathcal{G}}(\boldsymbol{p}, \boldsymbol{q})$, defined as

$$d_{\mathcal{G}}(\boldsymbol{p}, \boldsymbol{q}) := \inf_{\gamma \in \Gamma_{\boldsymbol{p},\boldsymbol{q}}} L_{\mathcal{G}}(\gamma), \tag{2.7}$$

where $\Gamma_{\boldsymbol{p},\boldsymbol{q}}$ is the set of piecewise smooth curves connecting $\boldsymbol{p}$ and $\boldsymbol{q}$.

Remark. *Such curves connecting $\boldsymbol{p}$ and $\boldsymbol{q}$ always exist, i.e. the set $\Gamma_{\boldsymbol{p},\boldsymbol{q}}$ is non-empty, as long as M is connected. This is the only case we consider.*

The curves γ that minimise Equation (2.7) are called (minimising) geodesics, generalising straight lines from Euclidean geometry. Like straight lines, they also have zero acceleration, and thus constant velocity, along the curve (in some parametrisation).

Definition 2.12 (Isometry)**.** Suppose $(M, \mathcal{G})$ and $(\widetilde{M}, \widetilde{\mathcal{G}})$ are two Riemannian manifolds. Then an isometry from $(M, \mathcal{G})$ to $(\widetilde{M}, \widetilde{\mathcal{G}})$ is a diffeomorphism $\psi : M \to \widetilde{M}$ such that for $\boldsymbol{p} \in M$ and tangent vectors $\boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{p}}M$ we have that

$$\mathcal{G}_{\boldsymbol{p}}(\boldsymbol{v}, \boldsymbol{w}) = \widetilde{\mathcal{G}}_{\psi(\boldsymbol{p})}(\psi_*(\boldsymbol{v}), \psi_*(\boldsymbol{w})). \tag{2.8}$$

This also means that $\|\boldsymbol{v}\|_{\mathcal{G}} = \|\psi_*(\boldsymbol{v})\|_{\widetilde{\mathcal{G}}}$.

For the final property of a Riemannian metric, we require an action of a group on the manifold. We start by recalling the definition of a group.

Definition 2.13 (Group)**.** A group $(G, \cdot)$ consists of a non-empty set G together with a group product $\cdot : G \times G \to G$ with the following properties:

- Associativity: For all $a, b, c \in G$: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- There exists an identity element $e \in G$ such that, for all $g \in G$; $e \cdot g = g = g \cdot e$.
- For all $g \in G$ there exists an inverse element $h \in G$: $g \cdot h = e = h \cdot g$. This is commonly written as $h = g^{-1}$.

The product operator $\cdot$ is usually omitted.

Definition 2.14 (Action of a group on a manifold)**.** A (left-)action of a group G on a manifold M is an action $\triangleright : G \times M \to M$ such that it is compatible with the action on the group: $g \triangleright (h \triangleright \boldsymbol{p}) = (gh) \triangleright \boldsymbol{p}$ for all $g, h \in G$ and $\boldsymbol{p} \in M$, and for the identity element $e \in G$: $e \triangleright \boldsymbol{p} = \boldsymbol{p}$.

Definition 2.15 (Invariant metric)**.** For a Riemannian manifold $(M, \mathcal{G})$, and a group G acting on M, the metric $\mathcal{G}$ is said to be invariant under G, or G-invariant, if for each $g \in G$ and $\boldsymbol{p} \in M$ the map given by the action $\boldsymbol{p} \mapsto g \triangleright \boldsymbol{p}$ is an isometry, Def. 2.12.

Remark. *Notice that if the manifold M itself carries a group structure (i.e. M is a Lie group, see also Section 2.3), the action on the manifold may be replaced with the group product.*

If the metric $\mathcal{G}$ is G-invariant, it follows that the lengths of curves are G-invariant, i.e. $L_{\mathcal{G}}(\gamma) = L_{\mathcal{G}}(g \triangleright \gamma)$, for all curves $\gamma : [0, 1] \to M$ and $g \in G$, such that all distances are G-invariant: $d_{\mathcal{G}}(\boldsymbol{p}, \boldsymbol{q}) = d_{\mathcal{G}}(g \triangleright \boldsymbol{p}, g \triangleright \boldsymbol{q})$ for $\boldsymbol{p}, \boldsymbol{q} \in M$ and any $g \in G$.

2.3 Lie theory

The following section will give an overview of relevant concepts from the fields of Lie Theory. This section is based on [19], to which the reader is referred for further explanation.

Lie theory considers Lie groups, which we define first.

Definition 2.16 (Lie Group)**.** A Lie group G is a smooth manifold (see Def. 2.1) that also has a group structure (see Def. 2.13) such that the group product and inversion map are smooth.

Note that since a Lie group is a manifold, all concepts from the previous sections still apply.

The simplest Lie groups are matrix Lie groups, i.e. Lie groups whose entries are matrices. The most important one is the General Linear group $\mathrm{GL}_n(\mathbb{R})$, consisting of all $n \times n$ invertible real matrices. Any other matrix Lie group is a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ (for some $n \in \mathbb{N}$).

Definition 2.17 (General Lie algebra)**.** A general Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear, antisymmetric Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that also satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \tag{2.9}$$

for all $X, Y, Z \in \mathfrak{g}$.

We are however interested in the Lie algebra that follows from a Lie group, which is given below.

Definition 2.18 (Lie algebra of a Lie group)**.** The Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space at the identity to G, i.e. $\mathfrak{g} = T_e G$, whose Lie bracket is defined, using the adjoint map ad defined below, as $[X, Y] := \mathrm{ad}_X Y$.

Elements of this Lie algebra are referred to as generators of the Lie group.

Definition 2.19 (Left- and right-multiplication operators and conjugation)**.** For a fixed $g \in G$ the left- and right-multiplication operators $L_g : G \to G$ and $R_g : G \to G$ are defined as

$$L_g h = gh, \quad R_g h = hg. \tag{2.10}$$

These can be used to define the conjugation operator $\mathrm{conj}_g : G \to G$ as

$$\mathrm{conj}_g h = ghg^{-1}, \quad \text{i.e.} \quad \mathrm{conj} = L_g \circ R_{g^{-1}}. \tag{2.11}$$

The push-forward at the identity e of the conjugation operator is the adjoint map:

$$\mathrm{Ad}_g = \left(\mathrm{conj}_g\right)_* : T_e G = \mathfrak{g} \to T_{\varphi_g(e)} G = \mathfrak{g}, \qquad \mathrm{Ad}_g = (L_g)_*(R_{g^{-1}})_*. \tag{2.12}$$

The mapping $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), g \mapsto \mathrm{Ad}_g$ is called the adjoint representation of G. The adjoint representation of the Lie algebra $\mathfrak{g}$ is given by the pushforward at the identity $\mathrm{ad} = (\mathrm{Ad})_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Remark. *For any matrix Lie group, the Lie bracket reduces to the commutator bracket, which is defined for (square) matrices A and B as*

$$[A, B] = AB - BA. \tag{2.13}$$

Definition 2.20 (Action of a Lie algebra on a manifold)**.** If the Lie group G has an action $\triangleright : G \times M \to M$ on a manifold M, the Lie algebra $\mathfrak{g} = T_e G$ also acts on the manifold. This action is given by the pushforward of the group action, i.e. consider any curve $\gamma : [0, 1] \to G$ with $\dot{\gamma}(0) = X$. Then this induced action is given by

$$X \triangleright \boldsymbol{p} := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\gamma(t) \triangleright \boldsymbol{p}), \tag{2.14}$$

for any $\boldsymbol{p} \in M$. In this same way we define the action of a Lie algebra on its corresponding Lie group.

Remark. *This action can also be thought of as a vector field, the so-called "fundamental" vector field, which is often identified with the generator itself.*

Definition 2.21 (Lie group homomorphism)**.** Let $(G, \cdot)$ and $(H, *)$ be two Lie groups. A map $\Phi : G \to H$ is called a Lie group homomorphism if

- Φ is a group homomorphism: for all $u, v \in G$: $\Phi(u \cdot v) = \Phi(u) * \Phi(v)$, and
- Φ is continuous.

If Φ is also bijective and its inverse Φ^{-1} is smooth, then it is called a Lie group isomorphism.

Definition 2.22 (Representation)**.** A representation of a Lie group G is a Lie group homomorphism $\Pi : G \to \mathrm{GL}(V)$ for a vector space V. This also gives rise to a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ of its Lie algebra $\mathfrak{g}$ via the pushforward at the identity $\pi = \Pi_*$.

A representation Π maps any arbitrary Lie group (or algebra) into a matrix Lie group (or algebra). This is because each element is mapped to a matrix and the group product is mapped to matrix multiplication, as Π is a homomorphism. Hence, also the action of the algebra on the group as well as the conjugation map (Def. 2.19) are given by matrix multiplication and the ad map or Lie bracket are given by the commutator bracket. Representing group and algebra elements as matrices also simplifies the exponential map defined below (Def. 2.24), as this is then given by the matrix exponential.

Definition 2.23 (Exponential curve)**.** Exponential curves on a Lie group G are defined as curves $\eta : \mathbb{R} \to G$ with the condition

$$\eta(s) \cdot \eta(t) = \eta(s+t), \quad s, t \in \mathbb{R}, \tag{2.15}$$

from which it follows that $\eta(0) = e \in G$ such that $\dot{\eta}(0) =: X \in \mathfrak{g}$. By differentiating the defining property in Equation (2.15) with respect to both s and t we see that

$$\eta(s)\, X = \dot{\eta}(s) \quad \text{and} \quad X\, \eta(t) = \dot{\eta}(t). \tag{2.16}$$

This means that the tangent vector to an exponential curve at any time $t \in \mathbb{R}$ is determined by its tangent vector at $t = 0$. Vice versa, we can also associate to each $X \in \mathfrak{g}$ an exponential curve η_X by imposing the property $\dot{\eta}_X(0) := X$.

Definition 2.24 (Exponential map on a Lie group)**.** With this we can define the exponential map for a Lie group G as the map

$$\exp : \mathfrak{g} \to G, \tag{2.17}$$

$$X \mapsto \exp(X) := \eta_X(1). \tag{2.18}$$

Remark. *In the case of a matrix Lie group, this reduces to the matrix exponential.*

Remark. *Hence, any exponential curve can be written as* $\eta_X(t) = \exp(tX) = e^{tX}$.

Definition 2.25 (Lie logarithm)**.** We define the logarithmic map as the inverse of the exponential map, as defined above, i.e. given an element $g \in G$, we define the corresponding generator $X = \log(g) \in \mathfrak{g}$ such that $e^X = g$.

Remark. *This logarithmic map is not necessarily (globally) unique, see also Remark 8 of [10].*

We are interested in Lie group actions on a manifold, which are group actions on the manifold, see Definition 2.14, that are also smooth. Their usefulness comes from the fact that the action of a Lie group on itself can also be seen in this way.

For the remainder of the section we are interested in lengths of exponential curves acting on the Riemannian manifold M with metric $\mathcal{G}$.

Lemma 2.1. *Let $X \in \mathfrak{g}$ be a generator of Lie group G, acting on manifold M, with exponential curve $\eta_X : t \mapsto e^{tX}$ and let $\boldsymbol{p}_0, \boldsymbol{p}_1 \in M$ with $e^X \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1$. Then if the metric $\mathcal{G}$ on M is invariant under the group action, see Def. 2.15, we have that (with the Lie algebra action as given in Definition 2.20)*

$$L_{\mathcal{G}}(\eta_X \triangleright \boldsymbol{p}_0) = \|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}. \tag{2.19}$$

Proof. Using Equation (2.16) and the definition of an invariant metric, Def. 2.15, we see that

$$L_{\mathcal{G}}(\eta_X \triangleright \boldsymbol{p}_0) = \int_0^1 \|\dot{\eta}_X(t) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}} \mathrm{d}t = \int_0^1 \|(\eta_X(t) X) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}} \mathrm{d}t = \int_0^1 \|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}} \mathrm{d}t = \|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}.$$

□

Definition 2.26 (Logarithmic distance)**.** For a Lie group G acting on a manifold M, with a G-invariant metric $\mathcal{G}$ on M, we define the logarithmic distance between $\boldsymbol{p}_0 \in M$ and $\boldsymbol{p}_1 \in M$ as

$$\rho_{\mathcal{G}}(\boldsymbol{p}_0, \boldsymbol{p}_1) = \inf_{e^X \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1} L_{\mathcal{G}}(\eta_X \triangleright \boldsymbol{p}_0) = \inf_{e^X \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1} \|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}, \tag{2.20}$$

using the previous Lemma 2.1.

Remark. *The logarithmic distance approximation on Lie group G with metric $\widetilde{\mathcal{G}}$ is also commonly defined using the Lie logarithm (Def. 2.25) as [10, 16]*

$$\rho_{\widetilde{\mathcal{G}}}(g, e) = \| \log g \|_{\widetilde{\mathcal{G}}}, \qquad g \in G. \tag{2.21}$$

Notice that one of the arguments is fixed at the identity $e \in G$.

Remark. *Instead of the true distance, Def. 2.11, which considers all curves between $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$, the logarithmic distance only considers exponential curves, and thus is an approximation.*

2.4 Homogeneous space

In this next section, we would like to identify the manifold M with equivalence classes of the Lie group G acting on it, in light of [16].

Definition 2.27 (Homogeneous space)**.** Consider a manifold M, acted upon by the Lie group G. If the Lie group acts on M transitively, i.e. for all elements $\boldsymbol{p}, \boldsymbol{q} \in M$ we have that $\boldsymbol{p} = g \triangleright \boldsymbol{q}$ for some $g \in G$, then it is referred to as a homogeneous space.

Remark. *Notice that this means that any pair of position-orientations can be rototranslated into each other. As such a rototranslation is in general not unique, we can define rototranslations to be equivalent if they map between the same pair of position-orientations. Then this homogeneous space can be interpreted as the group quotient of G under this equivalence relation. This is schematically visualised in Fig. 2.1 and will now be made more precise.*

Following this remark, for some (fixed) reference point $\boldsymbol{p}_0 \in M$ we define the equivalence relation $\sim$ as

$$g \sim h \Leftrightarrow g \triangleright \boldsymbol{p}_0 = h \triangleright \boldsymbol{p}_0, \quad g, h \in G. \tag{2.22}$$

It can be shown that the equivalence classes $[g]$ under this relation are given by the cosets gH, where H is the subgroup of G that fixes $\boldsymbol{p}_0$, i.e.

$$H = \mathrm{stab}_G(\boldsymbol{p}_0). \tag{2.23}$$

Then we can interpret M as the homogeneous space $M = G/H$ by identifying any $\boldsymbol{p} \in M$ with the equivalence class $[g] = gH \in G/H$ such that $g \triangleright \boldsymbol{p}_0 = \boldsymbol{p}$. We will also refer to this coset as a fibre.

On this homogeneous space we have the projection onto the fibres given by

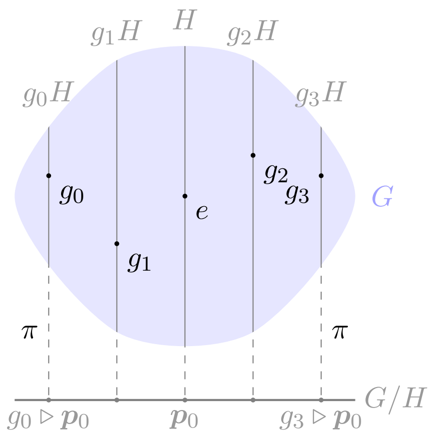


Figure 2.1: Schematic overview of the Lie group G with fibres $H = eH, g_0H, \cdots g_3H$ and the projection (π, dashed lines downwards) onto the group quotient G/H.

$$\pi : G \to G/H, \quad g \mapsto [g] = g \triangleright \boldsymbol{p}_0. \tag{2.24}$$

This is all visualised in Figure 2.1. Then we can make the following definition.

Definition 2.28 (Section)**.** Given a projection $\pi : G \to G/H$, a section of G is a map $\sigma : G/H \to G$ with $\pi \circ \sigma = \mathrm{Id}_{G/H}$[2].

[2]or equivalently $\sigma([g]) \in \pi^{-1}([g])$ or $\sigma([g]) \sim g$.

In order to find distances on the homogeneous space M, the metric defined on G must remain a Riemannian metric (Def. 2.8) on the equivalence classes. Such a metric is a *legal metric*:

Definition 2.29 (Legal metric)**.** A metric $\widetilde{\mathcal{G}}$ on a group G is legal if for some subgroup H of G:

$$\widetilde{\mathcal{G}}_{qg}((L_q)_*\dot{g},(L_g)_*\dot{g}) = \widetilde{\mathcal{G}}_g(\dot{g},\dot{g}) = \widetilde{\mathcal{G}}_{gh}((R_h)_*\dot{g},(R_h)_*\dot{g}), \tag{2.25}$$

for $g \in G, \dot{g} \in T_gG$, and $q \in G, h \in H$, i.e. $\mathcal{G}$ is invariant under left-multiplication by G and invariant under both left- and right-multiplication by H, i.e. $\mathrm{Ad}(H)$-invariant.

Remark. *Legal metrics are the left-invariant, i.e. invariant under left-multiplication, metrics on G which descend to Riemannian metrics on G/H [16].*

The distance and logarithmic distance on G descend to a (logarithmic) distance on G/H via

$$d_{\mathcal{G}}([g],[e]) = \min_{h\in H} d_{\widetilde{\mathcal{G}}}(gh,e) \quad \text{and} \quad \rho_{\mathcal{G}}([g],[e]) = \min_{h\in H} \rho_{\widetilde{\mathcal{G}}}(gh,e). \tag{2.26}$$

The following lemma discusses how a Riemannian metric on G/H can be made into a legal metric on G.

Lemma 2.2. *Consider a Lie group G acting on $M = G/H$, where $H = \mathrm{stab}_G(\boldsymbol{p}_0)$ for some $\boldsymbol{p}_0 \in M$, with metric $\mathcal{G} : TM \times TM \to \mathbb{R}$ on M that is G-invariant, and projection $\pi : G \to M$ given by $g \mapsto g \triangleright \boldsymbol{p}_0$, such that $\pi_* : TG \to TM$. Then the pseudometric $\widetilde{\mathcal{G}}$ on G given by*

$$\widetilde{\mathcal{G}} : TG \times TG \to \mathbb{R}, \quad \widetilde{\mathcal{G}}_g(u,v) := \mathcal{G}_{\pi(g)}(\pi_*u, \pi_*v) \tag{2.27}$$

for $u, v \in T_gG$, is legal.

Remark. *This is a pseudometric due to the fact that it is degenerate along H.*

Proof. We show that $\widetilde{\mathcal{G}}$ is left-invariant under G, and right-invariant under H: using that π and L_g commute for $g \in G$ and $\pi \circ R_h = \pi$ for $h \in H$ we have that

$$\pi_* g u = g \pi_* u, \quad \text{and} \quad \pi_*(R_h)_* = \pi_*$$

such that for any $g \in G$

$$\begin{aligned} \widetilde{\mathcal{G}}((L_g)_*u,(L_g)_*v) &= \mathcal{G}(\pi_* g u, \pi_* g v) \\ &= \mathcal{G}(g\pi_* u, g\pi_* v) \\ &= \mathcal{G}(\pi_* u, \pi_* v) = \widetilde{\mathcal{G}}(u,v), \quad \text{Left-invariant wrt } G \end{aligned}$$

and for any $h \in H$

$$\begin{aligned} \widetilde{\mathcal{G}}((R_h)_*u,(R_h)_*v) &= \mathcal{G}(\pi_*(R_h)_*u, \pi_*(R_h)_*v) \\ &= \mathcal{G}(\pi_* u, \pi_* v) = \widetilde{\mathcal{G}}(u,v). \quad \text{Right-invariant wrt H} \end{aligned}$$

Thus, $\widetilde{\mathcal{G}}$ is H-left- and right-invariant, so also $\mathrm{Ad}(H)$-invariant, and a legal metric. □

When for a metric $\mathcal{G}$ on $M = G/H$ the induced metric $\widetilde{\mathcal{G}}$ on G is defined via the previous Lemma 2.2 we have the following lemma about the norm of generators of G.

Lemma 2.3. *For the metric $\mathcal{G}$ on $M = G/H$ and induced pseudometric $\widetilde{\mathcal{G}}$ on G, we see that for the reference point $\boldsymbol{p}_0 \in M$ and generator $X \in \mathfrak{g}$ acting on $\boldsymbol{p}_0$ (Def. 2.20) we have that:*

$$\|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}} = \|X\|_{\widetilde{\mathcal{G}}}. \tag{2.28}$$

Proof. By writing out the claim, we see that

$$\|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}} = \sqrt{\mathcal{G}_{\boldsymbol{p}_0}(X \triangleright \boldsymbol{p}_0, X \triangleright \boldsymbol{p}_0)} = \sqrt{\mathcal{G}_{\pi(e)}(\pi_* X, \pi_* X)} = \sqrt{\widetilde{\mathcal{G}}_e(X, X)} = \|X\|_{\widetilde{\mathcal{G}}},$$

using that the action of the generators is the pushforward of the action of the group (see Def. 2.20). $\square$

Remark. *This would follow immediately if π is an isometry (Def. 2.12), but that is not the case as π is not a diffeomorphism.*

We finish this chapter with the following Lemma, expressing legality of a metric as a condition on the distance.

Lemma 2.4. *A metric $\widetilde{\mathcal{G}}$ on G is legal if and only if for the corresponding distance:*

$$d_{\widetilde{\mathcal{G}}}(qg_1, qg_2) = d_{\widetilde{\mathcal{G}}}(g_1, g_2) = d_{\widetilde{\mathcal{G}}}(g_1 h, g_2 h), \tag{2.29}$$

for all $q \in G, h \in H$ and $g_1, g_2 \in G$.

Proof. Let $g_1, g_2, q \in G$ and $h \in H$. First we assume that $\widetilde{\mathcal{G}}$ is legal. If γ is any curve connecting g_1 and g_2, then $\gamma' = q\gamma$ connects qg_1 and qg_2, and $\dot{\gamma}'(t) = (\dot{L_q\gamma})(t) = (L_q)_* \dot{\gamma}(t)$, such that, using Equations (2.6) and (2.7),

$$d_{\widetilde{\mathcal{G}}}(qg_1, qg_2) = \inf_{q\gamma \in \Gamma_{qg_1, qg_2}} \int_0^1 \sqrt{\widetilde{\mathcal{G}}_{q\gamma(t)}((L_q)_* \dot{\gamma}(t), (L_q)_* \dot{\gamma}(t))} \mathrm{d}t = \inf_{q\gamma} L_{\widetilde{\mathcal{G}}}(q\gamma). \tag{2.30}$$

Similarly, $\gamma'' = \gamma h$ connects $g_1 h$ and $g_2 h$, with $\dot{\gamma}''(t) = (\dot{R_h\gamma})(t) = (R_h)_* \dot{\gamma}(t)$, so

$$d_{\widetilde{\mathcal{G}}}(g_1 h, g_2 h) = \inf_{\gamma h \in \Gamma_{g_1 h, g_2 h}} \int_0^1 \sqrt{\widetilde{\mathcal{G}}_{\gamma(t)h}((R_h)_* \dot{\gamma}(t), (R_h)_* \dot{\gamma}(t))} \mathrm{d}t = \inf_{\gamma h} L_{\widetilde{\mathcal{G}}}(\gamma h). \tag{2.31}$$

It can be seen that if $\widetilde{\mathcal{G}}$ is legal the integrands of Equations (2.30) and (2.31) are equal to that in the calculation of $L_{\widetilde{\mathcal{G}}}(\gamma)$ at each $t \in [0, 1]$ such that $L_{\widetilde{\mathcal{G}}}(q\gamma) = L_{\widetilde{\mathcal{G}}}(\gamma) = L_{\widetilde{\mathcal{G}}}(\gamma h)$, and the distances are equal.

For the other direction, we assume that

$$d_{\widetilde{\mathcal{G}}}(qg_1, qg_2) = d_{\widetilde{\mathcal{G}}}(g_1, g_2) = d_{\widetilde{\mathcal{G}}}(g_1 h, g_2 h),$$

and for fixed $q, g_1 \in G, h \in H$ we consider the function

$$d_{\widetilde{\mathcal{G}}}(qg_1, \cdot) \circ L_q = d_{\widetilde{\mathcal{G}}}(g_1, \cdot) = d_{\widetilde{\mathcal{G}}}(g_1 h, \cdot) \circ R_h : G \to \mathbb{R}.$$

Taking any tangent vector $u \in T_{g_1} G$, and applying this (as a directional derivative) to all sides of this equation, yields

$$((L_q)_* u)(d_{\widetilde{\mathcal{G}}}(qg_1, \cdot)) = u(d_{\widetilde{\mathcal{G}}}(g_1, \cdot)) = ((R_h)_* u)(d_{\widetilde{\mathcal{G}}}(g_1 h, \cdot)). \tag{2.32}$$

Next, we make use of the result that the derivative of a distance function, is the metric $\widetilde{\mathcal{G}}$[3], i.e.

$$v(d_{\widetilde{\mathcal{G}}}(g, \cdot)) = \sqrt{\widetilde{\mathcal{G}}_g(v, v)} = \|v\|_{\widetilde{\mathcal{G}}}, \quad \forall g \in G, v \in T_g G,$$

such that equation (2.32) reduces to

$$\|(L_q)_* u\|_{\widetilde{\mathcal{G}}} = \|u\|_{\widetilde{\mathcal{G}}} = \|(R_h)_* u\|_{\widetilde{\mathcal{G}}},$$

i.e. the norm is preserved in the required way. This can be extended to all inner products by using the polarisation identity, such that $\widetilde{\mathcal{G}}$ is legal in the way of equation (2.25). $\square$

[3]This follows from the application of tangent vectors as directional derivatives: for $v = \dot{\gamma}(0)$ and a function f: $v(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))\big|_{t=0}$.

Chapter 3

Background

In the previous chapter we introduced the abstract theory of manifolds and Lie groups. In this chapter we introduce the specific manifold and Lie group on which we will be working, being the manifold of (d-dimensional) positions and orientations $\mathbb{M}_d$ and the group of rototranslations, also known as the special Euclidean group, $\mathrm{SE}(d)$ acting upon $\mathbb{M}_d$. This report focuses on the case $d = 3$, but for completeness they will be defined for general $d \in \mathbb{N}$.

3.1 Position-orientation space

Definition 3.1 (Position-orientation space)**.** The manifold $\mathbb{M}_d$ is the manifold of d-dimensional positions and orientations, which we view in the embedding $\mathbb{M}_d := \mathbb{R}^d \times S^{d-1} \subset \mathbb{R}^{d+d}$, where S^n is the n-dimensional sphere embedded in $n+1$-dimensional Euclidean space, so

$$\mathbb{M}_d := \mathbb{R}^d \times S^{d-1} = \{(\boldsymbol{x}, \boldsymbol{n}) \in \mathbb{R}^d \times \mathbb{R}^d \mid \|\boldsymbol{n}\| = 1\}. \tag{3.1}$$

Elements of this manifold will either be denoted by points $\boldsymbol{p} \in \mathbb{M}_d$ or pairs of position $\boldsymbol{x} \in \mathbb{R}^d$ and orientation $\boldsymbol{n} \in S^{d-1}$. The reference point $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0) \in \mathbb{M}_d$ is defined by $\boldsymbol{x}_0 = \boldsymbol{0}$ and $\boldsymbol{n}_0 = \boldsymbol{a}$ for any axis $\boldsymbol{a} \in S^{d-1}$. In the specific case $d = 3$, it is convention to take $\boldsymbol{a}$ to be the z-axis when considering the homogeneous space, although with our new parametrisation we are able to consider arbitrary $\boldsymbol{n}_0$.

For any pair of points in $\mathbb{M}_3$, we have the following definition.

Definition 3.2 (Coplanarity)**.** Two elements $(\boldsymbol{x}_0, \boldsymbol{n}_0), (\boldsymbol{x}_1, \boldsymbol{n}_1) \in \mathbb{M}_3$ are called coplanar if the vector connecting the two positions lies in the plane defined by the two orientations, or

$$\det(\boldsymbol{x}_1 - \boldsymbol{x}_0 \mid \boldsymbol{n}_0 \mid \boldsymbol{n}_1) = (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot (\boldsymbol{n}_0 \times \boldsymbol{n}_1) = 0. \tag{3.2}$$

This also means that we can write

$$\boldsymbol{x}_1 - \boldsymbol{x}_0 = \lambda \boldsymbol{n}_0 + \mu \boldsymbol{n}_1 \tag{3.3}$$

for constants $\lambda, \mu \in \mathbb{R}$. Discussion of how to find these constants can be found in Appendix B.

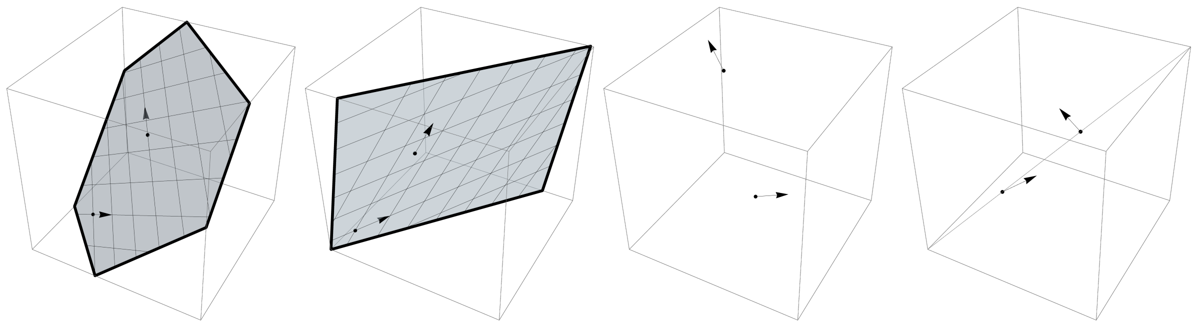
Examples of several pairs of position-orientations can be found in Figure 3.1.

Elements of $\mathbb{M}_d$ can also be written in a matrix representation as

$$\mathbb{M}_d \ni (\boldsymbol{x}, \boldsymbol{n}) \leftrightarrow \begin{pmatrix} \boldsymbol{x} & \boldsymbol{n} \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{(d+1)\times 2}. \tag{3.4}$$

The tangent space of S^{d-1} at some $\boldsymbol{n} \in S^{d-1}$ is $T_{\boldsymbol{n}} S^{d-1} = \{\dot{\boldsymbol{n}} \in \mathbb{R}^d \mid \dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 0\}$, where $\cdot$ is the usual inner product. Thus, at any $(\boldsymbol{x}, \boldsymbol{n}) \in \mathbb{M}_d$ the tangent space is given by

$$T_{(\boldsymbol{x},\boldsymbol{n})}\mathbb{M}_d = \{(\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}) \in \mathbb{R}^d \times \mathbb{R}^d \mid \dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 0\}. \tag{3.5}$$



(a) Coplanar (b) Coplanar (c) Non-coplanar (d) In mirror

Figure 3.1: Two examples of two position-orientations that are coplanar: $(\boldsymbol{x}_1-\boldsymbol{x}_0)\cdot(\boldsymbol{n}_0\times\boldsymbol{n}_1)=0$, one example of two position-orientation that are not coplanar and one example where the positions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are mirrored in the plane of $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$: $(\boldsymbol{x}_1-\boldsymbol{x}_0)\cdot(\boldsymbol{n}_0+\boldsymbol{n}_1)=0$.

3.2 Rototranslation group

Definition 3.3 (Special Euclidean group)**.** The *special Euclidean* group (in d dimensions) $\mathrm{SE}(d)$ is the Lie group consisting of translations and rotations (or explicitly rotation matrices) of $\mathbb{R}^d$:

$$\mathrm{SE}(d) := \{(\boldsymbol{t},\boldsymbol{R}) \in \mathbb{R}^d \times \mathbb{R}^{d\times d} \mid \boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}, \det \boldsymbol{R} = +1\}. \tag{3.6}$$

Elements of this group are called “rototranslations” and denoted either by elements $g \in \mathrm{SE}(d)$, or pairs of translation $\boldsymbol{t} \in \mathbb{R}^d$ and rotation $\boldsymbol{R} \in \mathbb{R}^{d\times d}$. In the case of $d = 3$, we write $G = \mathrm{SE}(3)$.

The Lie group $\mathrm{SE}(d)$ can be represented as a matrix Lie group under the representation

$$\mathrm{SE}(d) \ni (\boldsymbol{t},\boldsymbol{R}) \leftrightarrow \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0} & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}. \tag{3.7}$$

The group $\mathrm{SE}(d)$ acts on $(\boldsymbol{x},\boldsymbol{n}) \in \mathbb{M}_d$ as, denoted by $\triangleright$:

$$(\boldsymbol{t},\boldsymbol{R}) \triangleright (\boldsymbol{x},\boldsymbol{n}) := (\boldsymbol{t} + \boldsymbol{R}\boldsymbol{x}, \boldsymbol{R}\boldsymbol{n}), \tag{3.8}$$

which directly follows from multiplication of the matrix representations in Equations (3.4) and (3.7). Similarly it acts on elements of the tangent space as

$$T_{(\boldsymbol{x},\boldsymbol{n})}\mathbb{M}_d \ni (\dot{\boldsymbol{x}},\dot{\boldsymbol{n}}) \mapsto (\boldsymbol{R}\dot{\boldsymbol{x}},\boldsymbol{R}\dot{\boldsymbol{n}}) \in T_{(\boldsymbol{t},\boldsymbol{R})\triangleright(\boldsymbol{x},\boldsymbol{n})}\mathbb{M}_d. \tag{3.9}$$

The action of $\mathrm{SE}(d)$ on a function $f : \mathbb{M}_d \to \mathbb{R}$ defined on $\mathbb{M}_d$, is given by, for $g \in \mathrm{SE}(d)$,

$$(g \triangleright f)(\boldsymbol{p}) = f(g^{-1} \triangleright \boldsymbol{p}), \quad \forall\, \boldsymbol{p} \in \mathbb{M}_d. \tag{3.10}$$

The group operation of $\mathrm{SE}(d)$ on itself follows from the action on $\mathbb{M}_d$ via associativity, so

$$(\boldsymbol{t}_2,\boldsymbol{R}_2) \triangleright ((\boldsymbol{t}_1,\boldsymbol{R}_1) \triangleright (\boldsymbol{x},\boldsymbol{n})) = ((\boldsymbol{t}_2,\boldsymbol{R}_2)\cdot(\boldsymbol{t}_1,\boldsymbol{R}_1)) \triangleright (\boldsymbol{x},\boldsymbol{n}), \tag{3.11}$$

such that

$$(\boldsymbol{t}_2,\boldsymbol{R}_2)\cdot(\boldsymbol{t}_1,\boldsymbol{R}_1) = (\boldsymbol{t}_2 + \boldsymbol{R}_2\boldsymbol{t}_1, \boldsymbol{R}_2\boldsymbol{R}_1). \tag{3.12}$$

The identity element in this group is given by

$$e = (\boldsymbol{0},\boldsymbol{I}), \tag{3.13}$$

and the inverse operation is

$$(\boldsymbol{t},\boldsymbol{R})^{-1} = (-\boldsymbol{R}^{-1}\boldsymbol{t}, \boldsymbol{R}^{-1}). \tag{3.14}$$

Definition 3.4 (Lie algebra of the special Euclidean group)**.** The corresponding Lie algebra $\mathfrak{se}(d)$ consists of velocities $\boldsymbol{v}$ and angular velocity matrices $\boldsymbol{M}_\omega$:

$$\mathfrak{se}(d) = \{(\boldsymbol{v}, \boldsymbol{M}_\omega) \in \mathbb{R}^d \times \mathbb{R}^{d\times d} \mid \boldsymbol{M}_\omega^\top + \boldsymbol{M}_\omega = \boldsymbol{0}\}. \tag{3.15}$$

Here $\boldsymbol{M}_\omega$ is the generator of a rotation $\boldsymbol{R} = e^{\boldsymbol{M}_\omega}$ around an axis $\boldsymbol{\omega}$. The distinction between the two is visible in Figure 3.2. The angular velocity of the generator is encoded in the Euclidean length $\|\boldsymbol{\omega}\|$, meaning that the generated rotation $\boldsymbol{R}$ rotates an angle $\|\boldsymbol{\omega}\|$ around the given axis. For $d = 3$ the relation between the vector $\boldsymbol{\omega}$ and matrix $\boldsymbol{M}_\omega$ is given by $\boldsymbol{M}_\omega \boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{R}^3$. Following the matrix representation of $\mathrm{SE}(d)$ in Equation (3.7), this Lie algebra has the representation

$$\mathfrak{se}(d) \ni (\boldsymbol{v}, \boldsymbol{M}_\omega) \leftrightarrow \begin{pmatrix} \boldsymbol{M}_\omega & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}. \tag{3.16}$$

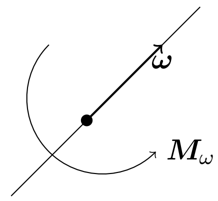


Figure 3.2: Distinction between generator $\boldsymbol{M}_\omega$ and rotation vector $\boldsymbol{\omega}$

The group action $\triangleright : \mathrm{SE}(d) \times \mathbb{M}_d \to \mathbb{M}_d$ induces an action $\triangleright : \mathfrak{se}(d) \times \mathbb{M}_d \to T\mathbb{M}_d$ (Definition 2.20) given by

$$(\boldsymbol{v}, \boldsymbol{M}_\omega) \triangleright (\boldsymbol{x}, \boldsymbol{n}) = (\boldsymbol{v} + \boldsymbol{M}_\omega \boldsymbol{x}, \boldsymbol{M}_\omega \boldsymbol{n}) \in T_{(\boldsymbol{x},\boldsymbol{n})}\mathbb{M}_d. \tag{3.17}$$

Definition 3.5 (Coplanar rototranslation)**.** A rototranslation $g \in G$ is said to act "coplanarly" if it maps the reference point $\boldsymbol{p}_0 \in \mathbb{M}_3$ to some $\boldsymbol{p}_1 \in \mathbb{M}_3$, where $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are coplanar, see Definition 3.2. For brevity this rototranslation may also be referred to as coplanar.

Definition 3.6. An element $A = (\boldsymbol{v}, \boldsymbol{\omega}) \in \mathfrak{g}$ generates a "pure rotation" if $\boldsymbol{v} \cdot \boldsymbol{\omega} = 0$.

This of course includes the case that $\boldsymbol{v} = \boldsymbol{0}$. The name is due to the fact that the generated rototranslation can be written as only a rotation and no translation, where the rotation may be around another axis (not necessarily through the origin).

Remark. *A pure rotation* $(\boldsymbol{v}, \boldsymbol{\omega}) \in \mathfrak{g}$ *generates a coplanar rototranslation* $\exp(\boldsymbol{v}, \boldsymbol{\omega})$ *with respect to the reference point* $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0)$ *if also* $\boldsymbol{\omega} \cdot \boldsymbol{n}_0 = 0$.

3.3 Lengths and distances on position-orientation space

In order to compute the length of any curve γ on $\mathbb{M}_d$ and the distance $d_{\mathcal{G}}(\boldsymbol{p}, \boldsymbol{q})$ between any two $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{M}_d$, we require a Riemannian metric $\mathcal{G}$ on the manifold $\mathbb{M}_d$, see also Section 2.3

For the length of any curve $\gamma : [0, 1] \to \mathbb{M}_d$ we want (the natural condition) that the length of the curve γ is equal to the length of the rototranslated curve $g \triangleright \gamma$ for any rototranslation $g \in \mathrm{SE}(d)$, i.e. the metric is invariant under $\mathrm{SE}(d)$, or $\mathrm{SE}(d)$-invariant, see Definition 2.15. Such $\mathrm{SE}(d)$-invariant metrics on $\mathbb{M}_d$ are given [17], for $(\boldsymbol{x}, \boldsymbol{n}) \in \mathbb{M}_d$ and $(\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}) \in T_{(\boldsymbol{x},\boldsymbol{n})}\mathbb{M}_d$, by

$$\mathcal{G}_{(\boldsymbol{x},\boldsymbol{n})}((\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}), (\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}})) = w_1 |\dot{\boldsymbol{x}} \cdot \boldsymbol{n}|^2 + w_2 \|\dot{\boldsymbol{x}} \wedge \boldsymbol{n}\|^2 + w_3 \|\dot{\boldsymbol{n}}\|^2, \tag{3.18}$$

where $w_1, w_2, w_3 \geqslant 0$ are constants weighing the three types of motion: the motion parallel and orthogonal to the orientation $\boldsymbol{n}$ as well as angular motion. Here the symbol $\wedge$ denotes the wedge product of two vectors and, as we are only interested in three dimensions, we replace this by the standard cross product $\times$.

Remark. *Equation (3.18) does not constitute all SE(d)-invariant metrics, only all diagonal metrics. As we will only consider diagonal metrics, we will write all metrics in this way.*

The (spatial) anisotropy of such a metric is defined as

$$\zeta := \frac{w_2}{w_1} \tag{3.19}$$

and the metrics in Eq. (3.18) are valid only for $1 \leqslant \zeta \leqslant 4$, as otherwise a different, sub-Riemannian metric is needed. In the spatially isotropic case given by $\zeta = 1$, such that we write $w_1 = w_2 = w$, the metric reduces to

$$\mathcal{G}_{(\boldsymbol{x},\boldsymbol{n})}((\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}), (\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}})) = w\|\dot{\boldsymbol{x}}\|^2 + w_3\|\dot{\boldsymbol{n}}\|^2. \tag{3.20}$$

With these metrics we can define both the SE(d)-invariant true distance $d_{\mathcal{G}}$ as well as the (also invariant) logarithmic distance approximation $\rho_{\mathcal{G}}$ on $\mathbb{M}_d$, although the latter can, see Definition 2.26, be simplified as

$$\rho_{\mathcal{G}}(\boldsymbol{p}_0, \boldsymbol{p}_1) := \inf_{e^X \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1} L_{\mathcal{G}}(\eta_X \triangleright \boldsymbol{p}_0) = \inf_{e^X \triangleright \boldsymbol{p}_0 = \boldsymbol{p}_1} \|X \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}. \tag{3.21}$$

3.4 Homogeneous space

For the reference point $\boldsymbol{p}_0 = (\mathbf{0}, \boldsymbol{a}) \in \mathbb{M}_3$, we can write $H = \text{stab}_G(\boldsymbol{p}_0)$ and $\mathbb{M}_3 = G/H$ for $G = \text{SE}(3)$ as in Section 2.4. Then we have that H is the set of rotations around the axis $\boldsymbol{a}$, i.e. H is isomorphic to the Lie group SO(2) of 2D rotations.

Given a legal metric $\widetilde{\mathcal{G}}$ on G and fibre $[g] \in G/H$, we would like to know which rototranslation $g \in [g]$ minimises the Riemannian distance from $[e] = \boldsymbol{p}_0 \in G/H$ to $[g]$. This is given by the section (Def. 2.28) σ_d, defined as

$$\sigma_d([g]) := \arg\min_{h \in [g]} d_{\widetilde{\mathcal{G}}}(h, e). \tag{3.22}$$

As this is expensive to compute [14, 15], we are interested in approximating this distance, for example by using the logarithmic norm approximation $\rho_{\widetilde{\mathcal{G}}}$:

$$\sigma_\rho([g]) = \arg\min_{h \in [g]} \rho_{\widetilde{\mathcal{G}}}(h, e). \tag{3.23}$$

As this section is quite complicated, we are interested in approximation by a simpler section, see also introduction (Chapter 1). One such section is the section σ_p [16], which is given in the Euler angle parametrisation (see e.g. [2]) by the simple condition that $\alpha = -\gamma$, so

$$\sigma_p([\boldsymbol{t}, \boldsymbol{R}_{\boldsymbol{e}_z,\gamma}\boldsymbol{R}_{\boldsymbol{e}_y,\beta}\boldsymbol{R}_{\boldsymbol{e}_z,\alpha}]) = (\boldsymbol{t}, \boldsymbol{R}_{\boldsymbol{e}_z,\gamma}\boldsymbol{R}_{\boldsymbol{e}_y,\beta}\boldsymbol{R}_{\boldsymbol{e}_z,-\gamma}). \tag{3.24}$$

It can be shown [16] that this section σ_p is equal to σ_{MAV}, the section generated by the MAV generator, or generator with Minimum Angular Velocity:

$$\sigma_{\text{MAV}}([g]) = \arg\min_{\substack{\exp(X) \in [g], \\ X = (\boldsymbol{v}, \boldsymbol{M}_\omega) \in \mathfrak{g}}} \|\boldsymbol{\omega}\|, \tag{3.25}$$

such that we will write σ_{MAV} everywhere. When considering rototranslations between $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ we may write the MAV generator as $M(\boldsymbol{p}_0, \boldsymbol{p}_1)$, such that $\sigma_{\text{MAV}}([g]) = \exp(M(\boldsymbol{p}_0, \boldsymbol{p}_1))$ for $\boldsymbol{p}_1 = g \triangleright \boldsymbol{p}_0$.

Remark. *In Equation (3.25) we can use the Euclidean norm of the angular velocity vector* $\boldsymbol{\omega}$ *associated to* $\boldsymbol{M}_\omega$ *since* $\widetilde{\mathcal{G}}$ *is legal and thereby*

$$\|(\mathbf{0}, \boldsymbol{M}_\omega)\|^2_{\widetilde{\mathcal{G}}} = w_3\|\boldsymbol{\omega}\|^2 \tag{3.26}$$

and the constant value of $w_3 > 0$ *does not matter for the optimisation.*

For these sections we also have the following chain of inequalities, Proposition 1 of [16].

Lemma 3.1. *For the identity $e \in G$ and fibres $[g] = gH \in G/H$, we have that*

$$\rho_{\widetilde{\mathcal{G}}}(\sigma_{\mathrm{MAV}}([g]), e) \geqslant \rho_{\widetilde{\mathcal{G}}}(\sigma_{\rho}([g]), e) \geqslant d_{\widetilde{\mathcal{G}}}(\sigma_{d}([g]), e) \tag{3.27}$$

Proof. Since σ_ρ is the section that minimises $\rho_{\widetilde{\mathcal{G}}}$, we have that

$$\rho_{\widetilde{\mathcal{G}}}(\sigma_{\rho}([g]), e) = \min_{h \in [g]} \rho_{\widetilde{\mathcal{G}}}(h, e) \leqslant \rho_{\widetilde{\mathcal{G}}}(\sigma([g]), e)$$

for any section σ, in particular for $\sigma = \sigma_p = \sigma_{\mathrm{MAV}}$. By their definition, we also know that $\rho_{\widetilde{\mathcal{G}}}(h, e) \geqslant d_{\widetilde{\mathcal{G}}}(h, e)$ for all $h \in G$, such that we can conclude

$$\rho_{\widetilde{\mathcal{G}}}(\sigma_{\mathrm{MAV}}([g]), e) \geqslant \rho_{\widetilde{\mathcal{G}}}(\sigma_{\rho}([g]), e) = \min_{h \in [g]} \rho_{\widetilde{\mathcal{G}}}(h, e) \geqslant \min_{h \in [g]} d_{\widetilde{\mathcal{G}}}(h, e) = d_{\widetilde{\mathcal{G}}}(\sigma_{d_{\widetilde{\mathcal{G}}}}([g]), e).$$

□

Remark. *The estimate in Lemma 3.1 holds with equality when restricting $[g]$ to $\{\mathbf{0}\} \times \mathrm{SO}(3)/H$ as proven in [16, Prop. 1]. This proves that σ_p selects the rototranslation whose generator has minimal angular velocity, i.e. the MAV-generator, see Remark 2.*

Chapter 4

Coordinates and the norm

In this chapter we will discuss the logarithmic coordinates for elements of the Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$ and compute explicitly the exponential and logarithm maps between $\mathfrak{g}$ and $G = \mathrm{SE}(3)$. This logarithmic map is needed to compute the logarithmic distance ρ as in Eq. (2.21). Using an example, we show how difficult this is to optimise. Afterwards we introduce the new coordinates based on screw transformations and show how this simplifies the computations.

4.1 Logarithmic coordinates

Generators $X \in \mathfrak{g}$ are of the form

$$X = \sum_{i=1}^{6} c^i A_i = \boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)}, \tag{4.1}$$

where $\{A_i\}_{i=1}^6$ is the basis of $\mathfrak{g}$. In the interpretation as tangent vectors, A_1, A_2, A_3, also collected in $\boldsymbol{A}_{(1)}$, are related to translation along the x-,y- and z-axis respectively, and A_4, A_5, A_6, or $\boldsymbol{A}_{(2)}$ are rotations around the x-,y- and z-axis respectively, i.e. [16]

$$\{A_i\}_{i=1}^6 = \{\, \partial_x|_e \,, \partial_y|_e \,, \partial_z|_e \,, \partial_{\widetilde{\gamma}} \boldsymbol{R}_{\boldsymbol{e}_x,\widetilde{\gamma}}\big|_{\widetilde{\gamma}=0} \,, \partial_{\widetilde{\beta}} \boldsymbol{R}_{\boldsymbol{e}_y,\widetilde{\beta}}\Big|_{\widetilde{\beta}=0} \,, \partial_{\widetilde{\alpha}} \boldsymbol{R}_{\boldsymbol{e}_z,\widetilde{\alpha}}\big|_{\widetilde{\alpha}=0} \}. \tag{4.2}$$

When using this, we always set $\boldsymbol{n}_0 = \boldsymbol{e}_z$, such that A_3 and A_6 are the translation along and rotation about $\boldsymbol{n}_0$ respectively. When writing $X = (\boldsymbol{v}, \boldsymbol{M}_\omega)$, we have that $\boldsymbol{c}^{(1)} = \boldsymbol{v}$ and $\boldsymbol{c}^{(2)} = \boldsymbol{\omega}$, i.e. $X = \boldsymbol{v} \cdot \boldsymbol{A}_1 + \boldsymbol{\omega} \cdot \boldsymbol{A}_2$.

Using Equation (4.1) and the exponential map, Def. 2.24, elements $g \in G$ can be written as

$$g = \exp\left(\sum_{i=1}^{6} c^i A_i\right) = e^{\boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)}} = e^{(\boldsymbol{c}^{(1)}, \boldsymbol{M}_{c^{(2)}})}. \tag{4.3}$$

The simple section σ_{MAV}, Eq. (3.24), is expressed by the condition $c^6 = 0$ (Eq. 4 of [16]) and elements h of the subgroup H are expressed by the condition that $h = e^{c^6 A_6}$, i.e. these are generated solely by the rotation basis element A_6.

We have the following Lemma and Corollary:

Lemma 4.1 (Rotating a generator)**.** *For generator $(\boldsymbol{v}, \boldsymbol{M}_\omega) \in \mathfrak{g}$ and rotation $(\boldsymbol{0}, \boldsymbol{R}) \in G$, we have that*

$$\mathrm{Ad}_{(\boldsymbol{0},\boldsymbol{R})}(\boldsymbol{v}, \boldsymbol{M}_\omega) := (\boldsymbol{0}, \boldsymbol{R})(\boldsymbol{v}, \boldsymbol{M}_\omega)(\boldsymbol{0}, \boldsymbol{R})^{-1} = (\boldsymbol{R}\boldsymbol{v}, \boldsymbol{M}_{\boldsymbol{R}\boldsymbol{\omega}}), \tag{4.4}$$

or equivalently

$$\mathrm{conj}_{(\boldsymbol{0},\boldsymbol{R})} e^{(\boldsymbol{v},\boldsymbol{M}_\omega)} := (\boldsymbol{0}, \boldsymbol{R}) e^{(\boldsymbol{v},\boldsymbol{M}_\omega)} (\boldsymbol{0}, \boldsymbol{R})^{-1} = e^{(\boldsymbol{R}\boldsymbol{v}, \boldsymbol{M}_{\boldsymbol{R}\boldsymbol{\omega}})}. \tag{4.5}$$

Remark. *This states that conjugating a generator with a rotation has the same effect as rotating the generator.*

Proof. The equivalence comes from the fact that for $g \in G$ and $X \in \mathfrak{g}$, we have in general that

$$\mathrm{conj}_g e^X = e^{\mathrm{Ad}_g X}.$$

Then, since Ad and the matrix representation commute, see the discussion under Def. 2.22, we know that the matrix representation of the left-hand side of Equation (4.4) is given by

$$\begin{pmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{M}_\omega & \boldsymbol{v} \\ \boldsymbol{0} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}\boldsymbol{M}_\omega \boldsymbol{R}^{-1} & \boldsymbol{R}\boldsymbol{v} \\ \boldsymbol{0} & 0 \end{pmatrix},$$

such that this is equal to the right-hand side of Equation (4.4) when $\boldsymbol{R}\boldsymbol{M}_\omega\boldsymbol{R}^{-1} = \boldsymbol{M}_{\boldsymbol{R}\omega}$, which we show by applying to some $\boldsymbol{x} \in \mathbb{R}^3$:

$$\boldsymbol{R}\boldsymbol{M}_\omega\boldsymbol{R}^{-1}\boldsymbol{x} = \boldsymbol{R}(\boldsymbol{\omega} \times (\boldsymbol{R}^{-1}\boldsymbol{x})) = (\boldsymbol{R}\boldsymbol{\omega}) \times (\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{x}) = \boldsymbol{M}_{\boldsymbol{R}\omega}\boldsymbol{x}.$$

Thus, conjugating a generator with a sole rotation has the same effect as rotating the generator. □

Corollary 4.1. *For any $X = \boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)} \in \mathfrak{g}$ and $t \geqslant 0$, we have that*

$$e^{tA_6} \exp\left(\boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)}\right) e^{-tA_6} = \exp\left(\boldsymbol{R}_{\boldsymbol{e}_z,t}\boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{R}_{\boldsymbol{e}_z,t}\boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)}\right). \tag{4.6}$$

Proof. This follows immediately from the previous Lemma, using that

$$e^{tA_6} = (\boldsymbol{0}, \boldsymbol{R}_{\boldsymbol{e}_z,t}) \in G$$

is the pure rotation around the z-axis for an angle of t radians. □

We have seen that a pure rotation $(\boldsymbol{v}, \boldsymbol{\omega}) \in \mathfrak{g}$ generates a coplanar rototranslation (Def. 3.5) under the condition that $\boldsymbol{\omega} \cdot \boldsymbol{n}_0 = 0$, which is equivalent with saying that $c^6 = 0$. Due to these restrictions, there are only two degrees of freedom and the axes can be defined such that $\boldsymbol{v}$ and $\boldsymbol{n}_0$ are parallel, after which $\boldsymbol{\omega}$ is parallel to one of the other basis axes either by defining the axis in that way or by using Corollary 4.1).

4.1.1 Computing the exponential and logarithm

Now we will, in general, explicitly compute the exponential and logarithmic map between G and $\mathfrak{g}$, starting with the exponential. Due to the matrix representation of both the Lie group and algebra, the exponential map is simply given by the matrix exponential of those matrices. We follow the steps in section 5.1 of [2]. Let $X = \boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)} = (\boldsymbol{v}, \boldsymbol{M}_\omega) \in \mathfrak{g}$. The matrix exponential can be computed to be, for $t \geqslant 0$:

$$e^{tX} = \begin{pmatrix} e^{t\boldsymbol{M}_\omega} & t\left(\sum_{k=0}^{\infty} \frac{(t\boldsymbol{M}_\omega)^k}{(k+1)!}\right)\boldsymbol{v} \\ \boldsymbol{0} & 1 \end{pmatrix}, \tag{4.7}$$

and we are interested in knowing to which $(\boldsymbol{x}, \boldsymbol{R}) \in G$ this corresponds[1].

[1]Recall that usually the translation vector is denoted by $\boldsymbol{t}$, but here we use the vector $\boldsymbol{x}$ due to following [2]. This also leads to less confusion with the parameter t.

Upon writing the angular velocity as $\omega = \|\boldsymbol{\omega}\| = \|\boldsymbol{c}^{(2)}\| = \sqrt{|c^4|^2 + |c^5|^2 + |c^6|^2}$, we have the following recurrence relation:

$$\boldsymbol{M}_\omega^3 = -\omega^2 \boldsymbol{M}_\omega, \tag{4.8}$$

such that we can compute

$$\sum_{k=0}^{\infty} \frac{(t\boldsymbol{M}_\omega)^k}{(k+1)!} = \boldsymbol{I} + \frac{1-\cos\omega t}{\omega^2 t}\boldsymbol{M}_\omega + \frac{\omega t - \sin\omega t}{\omega^3 t}\boldsymbol{M}_\omega^2 \quad \text{and}$$
$$e^{t\boldsymbol{M}_\omega} = \boldsymbol{I} + \frac{\sin\omega t}{\omega}\boldsymbol{M}_\omega + \frac{1-\cos\omega t}{\omega^2}\boldsymbol{M}_\omega^2.$$

This means that the exponential curve e^{tX} maps the generator X to the rototranslation $e^{tX} = (\boldsymbol{x}, \boldsymbol{R})$ with translation vector and rotation matrix given by

$$\boldsymbol{x} = t\boldsymbol{v} + \frac{1-\cos\omega t}{\omega^2}\boldsymbol{M}_\omega \boldsymbol{v} + \frac{\omega t - \sin\omega t}{\omega^3}\boldsymbol{M}_\omega^2 \boldsymbol{v}, \tag{4.9a}$$
$$\boldsymbol{R} = \boldsymbol{I} + \frac{\sin\omega t}{\omega}\boldsymbol{M}_\omega + \frac{1-\cos\omega t}{\omega^2}\boldsymbol{M}_\omega^2. \tag{4.9b}$$

In the case that there is no rotation generator, i.e. $\boldsymbol{\omega} = \boldsymbol{0}$ such that $\boldsymbol{M}_\omega = \boldsymbol{0}$ and $\omega = 0$, we see from the previous set of equations that $\boldsymbol{x} = t\boldsymbol{v}$ and $\boldsymbol{R} = \boldsymbol{I}$, i.e. there is no rotation, as is to be expected.

Hence, we have shown the exponential map from the Lie algebra to the Lie group. We are, however, also interested in its inverse: Given an element $(\boldsymbol{x}, \boldsymbol{R}) \in G$, what is the corresponding generator $X \in \mathfrak{g}$ such that $e^X = (\boldsymbol{x}, \boldsymbol{R})$ (notice that $t = 1$). This mapping is given by the logarithm: $X = \log(\boldsymbol{x}, \boldsymbol{R})$ and will be derived using the expressions above. We can notice that the matrix $\boldsymbol{M}_\omega$ is anti-symmetric, such that $\boldsymbol{M}_\omega^2$ is symmetric. This means that if $\boldsymbol{R}$ is the rotation matrix resulting from an exponential curve, we have that

$$\boldsymbol{R} - \boldsymbol{R}^\top = 2\frac{\sin\omega}{\omega}\boldsymbol{M}_\omega. \tag{4.10}$$

Any rotation matrix $\boldsymbol{R}$ can be expressed in Euler angles as

$$\begin{aligned}\boldsymbol{R} = \boldsymbol{R}_{\gamma,\beta,\alpha} &= \boldsymbol{R}_{\boldsymbol{e}_z,\gamma}\boldsymbol{R}_{\boldsymbol{e}_y,\beta}\boldsymbol{R}_{\boldsymbol{e}_z,\alpha} \\ &= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

such that upon comparing indices, we can see that

$$\begin{cases} \sin\beta(\sin\alpha - \sin\gamma) & = 2\frac{\sin\omega}{\omega}c^4 \\ \sin\beta(\cos\alpha + \cos\gamma) & = 2\frac{\sin\omega}{\omega}c^5 \\ 2\cos^2\frac{\beta}{2}\sin(\alpha+\gamma) & = 2\frac{\sin\omega}{\omega}c^6 \end{cases}, \tag{4.11}$$

which determines the coefficients c^4, c^5, c^6 in $\boldsymbol{c}^{(2)}$. Via squaring and summing we can also determine their norm ω implicitly as

$$\sin^2\omega = \sin^2\beta\cos^2\frac{\alpha+\gamma}{2} + \cos^4\frac{\beta}{2}\sin^2(\alpha+\gamma). \tag{4.12}$$

For the full logarithm, also the coefficients $\boldsymbol{c}^{(1)}$ have to be determined. Via the exponential map, these are related to the resulting translation using Equation (4.9a), i.e.

$$\boldsymbol{c}^{(1)} = \left(\boldsymbol{I} + \frac{1-\cos\omega}{\omega^2}\boldsymbol{M}_\omega + \frac{\omega - \sin\omega}{\omega^3}\boldsymbol{M}_\omega^2\right)^{-1}\boldsymbol{x}. \tag{4.13}$$

This matrix inverse can be written in the form $a\boldsymbol{I} + b\boldsymbol{M}_\omega + c\boldsymbol{M}_\omega^2$, for some constants $a, b, c \in \mathbb{R}$ possibly dependent on ω, using the recurrence relation in Eq. (4.8). Multiplying this out leads to the equation

$$\boldsymbol{c}^{(1)} = \left(a\boldsymbol{I} + b\boldsymbol{M}_\omega + c\boldsymbol{M}_\omega^2\right)\boldsymbol{x} = \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{M}_\omega + \frac{1 - \frac{\omega}{2}\cot\frac{q}{2}}{q^2}\boldsymbol{M}_\omega^2\right)\boldsymbol{x}, \tag{4.14}$$

where $\cot z = \frac{1}{\tan z}$.

Example 4.1.1 (Logarithm)**.** *In order to clarify the procedure outlined above, we apply the logarithm to a specific element of G. Let $X = d^3 A_3 + d^4 A_4 \in \mathfrak{g}$ and $g = e^X \in G$. Since in this case the logarithm of g is trivial, we also multiply this with an element $h = \exp(d^6 A_6) \in H$ to create the equivalence class $[g]$ and take the logarithm of this.*

For g we have that $\boldsymbol{v} = (0, 0, d^3)$, $\boldsymbol{\omega} = (d^4, 0, 0)$ and $\omega_g = d^4$, such that the rotation matrix and translation vector are given by

$$\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{e}_x, d^4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos d^4 & -\sin d^4 \\ 0 & \sin d^4 & \cos d^4 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{x} = \frac{d^3}{d^4}\begin{pmatrix} 0 \\ \cos d^4 - 1 \\ \sin d^4 \end{pmatrix}. \tag{4.15}$$

For h on the other hand we have that $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{\omega} = (0, 0, d^6)$, so $\omega_h = d^6$. This means that there is no translation, and the rotation matrix is given by $\boldsymbol{R} = \boldsymbol{R}_{\boldsymbol{e}_z, d^6}$.

Then the multiplication between g and h is given by

$$gh = (\boldsymbol{x}, \boldsymbol{R}_{\boldsymbol{e}_x, d^4}) \cdot (\boldsymbol{0}, \boldsymbol{R}_{\boldsymbol{e}_z, d^6}) = (\boldsymbol{x}, \boldsymbol{R}_{\boldsymbol{e}_x, d^4}\boldsymbol{R}_{\boldsymbol{e}_z, d^6}), \tag{4.16}$$

whose rotation matrix has Euler angles $\alpha = d^6 + \frac{\pi}{2}, \beta = d^4$ and $\gamma = -\frac{\pi}{2}$. This means that $\log(gh)$ *has angular coefficients*

$$\begin{cases} c^4 &= \frac{\omega}{\sin\omega}\sin d^4 \cos^2\frac{d^6}{2} \\ c^5 &= -\frac{\omega}{2\sin\omega}\sin d^4 \sin d^6 \\ c^6 &= \frac{\omega}{\sin\omega}\cos^2\frac{d^4}{2}\sin d^6 \end{cases} \tag{4.17}$$

with

$$\sin^2\omega = \sin^2 d^4 \cos^2\frac{d^6}{2} + \cos^4\frac{d^4}{2}\sin^2 d^6. \tag{4.18}$$

The other coefficients are given by

$$\begin{aligned} \boldsymbol{c}^{(1)} &= \left(\boldsymbol{I} - \frac{1}{2}\boldsymbol{M}_\omega + \nu\boldsymbol{M}_\omega^2\right)\boldsymbol{x} \\ &= \frac{d^3}{d^4}\begin{pmatrix} 1 - \nu\left((c^5)^2 + (c^6)^2\right) & \nu c^4 c^5 + \frac{c^6}{2} & \nu c^4 c^6 - \frac{c^5}{2} \\ \nu c^4 c^5 - \frac{c^6}{2} & 1 - \nu\left((c^4)^2 + (c^6)^2\right) & \nu c^5 c^6 + \frac{c^4}{2} \\ \nu c^4 c^6 + \frac{c^5}{2} & \nu c^5 c^6 - \frac{c^4}{2} & 1 - \nu\left((c^4)^2 + (c^5)^2\right) \end{pmatrix}\begin{pmatrix} 0 \\ \cos d^4 - 1 \\ \sin d^4 \end{pmatrix}, \end{aligned}$$

with $\nu = \frac{1 - \frac{\omega}{2}\cot\frac{\omega}{2}}{\omega^2}$.

It should be clear from this example that the logarithm is not easy to compute, in particular since we want to compute the logarithmic distance

$$\rho_{\mathcal{G}}([g], [e]) = \inf_{h \in H} \rho_{\widetilde{\mathcal{G}}}(gh, e) = \inf_{h \in H} \|\log gh\|_{\widetilde{\mathcal{G}}} = \inf_{d^6 \in \mathbb{R}} \|\log e^{d^3 A_3 + d^4 A_4} e^{d^6 A_6}\|_{\widetilde{\mathcal{G}}}, \tag{4.19}$$

or answer the question whether the minimiser here is given by $d^6 = 0$, corresponding to the simple section σ_{MAV}.

4.2 A new perspective: screw motion parametrisation

In the remainder of this chapter, we employ a new view of rototranslations as screw transformations, which is possible by Chasles' Theorem, see e.g. Section 2.1 of [20]. We start by describing what such a screw transformation is, see also e.g. [17].

Definition 4.1 (Screw transformation)**.** A screw transformation consists of a rotation around a rotation axis (not necessarily passing through the origin) as well as a translation in the same direction as the rotation axis.

As given in Equation (3.8), an arbitrary rototranslation $g = (\boldsymbol{s}, \boldsymbol{R}) \in G$ maps any $\boldsymbol{x} \in \mathbb{R}^3$ as

$$g \triangleright \boldsymbol{x} = \boldsymbol{s} + \boldsymbol{R}\boldsymbol{x}, \tag{4.20}$$

which we can write as a screw transformation as

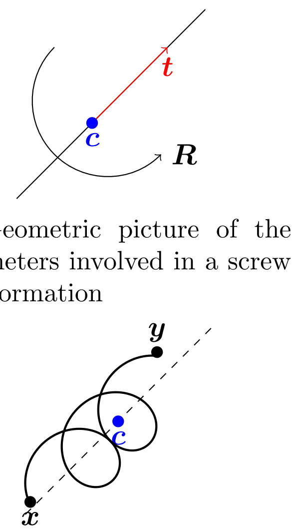
$$g \triangleright \boldsymbol{x} = \boldsymbol{c} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{c}) + \boldsymbol{t} \tag{4.21}$$

with $\boldsymbol{R}\boldsymbol{t} = \boldsymbol{t}$. In this way we have that $\boldsymbol{c}$ is a point on the rotation axis, while $\boldsymbol{t}$ is the remaining translation vector along the axis of rotation, as illustrated in Figure 4.1a. This can also be written as a screw motion, i.e. a function of $s \in [0, 1]$ as

$$e^{s\chi} = g_s : \boldsymbol{x} \mapsto \boldsymbol{c} + e^{s\boldsymbol{M}_\omega}(\boldsymbol{x} - \boldsymbol{c}) + s\boldsymbol{t}, \tag{4.22}$$

where $e^{\boldsymbol{M}_\omega} = \boldsymbol{R}$ and with generator $\chi \in \mathfrak{g}$[2].

This is a smooth transformation for $s \in [0, 1]$, and is illustrated in Figure 4.1b. Recall the distinction between the rotation generator $\boldsymbol{M}_\omega$ and the corresponding rotation vector $\boldsymbol{\omega}$, see the discussion under Definition 3.4 and Figure 3.2. This means that the generator is given by



(a) Geometric picture of the parameters involved in a screw transformation

(b) Screw motion from $\boldsymbol{x}$ to $\boldsymbol{y}$

Figure 4.1

$$\chi = (-\boldsymbol{M}_\omega \boldsymbol{c} + \boldsymbol{t}, \boldsymbol{M}_\omega). \tag{4.23}$$

In the remainder of this section, we assume (fixed) points $\boldsymbol{p}_0, \boldsymbol{p}_1 \in \mathbb{M}_3$ and construct the screw transformations mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$. We will see these can be indexed by a single parameter φ, such that in general for e.g. the generator we have the dependence $\chi = \chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi)$, although for simplicity we suppress the $\boldsymbol{p}_0, \boldsymbol{p}_1$-dependence.

4.2.1 Orientation parametrisation of the screw motion

Writing $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0)$ and $\boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1)$, we start with the effect of a screw transformation on the orientations, i.e. the matrix group of rotations SO(3) acting on $S^2 \subset \mathbb{R}^3$. Assuming the orientations $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are not colinear[3], we can orient them on the equator, as in Figure 4.2a. The situation that $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are colinear is handled in the later Section 4.5.

For every element $\boldsymbol{k}$ in the bisector, or mirror, plane between $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$, there exists a rotation around $\boldsymbol{k}$ mapping $\boldsymbol{n}_0$ to $\boldsymbol{n}_1$. Vice versa, we have that any such rotation is about an axis $\boldsymbol{k}$ in this bisector plane as the distance from the axis to both orientations must be equal.

We will index these elements by the polar angle $\varphi \in [-\pi, \pi)$, i.e. we write

$$\boldsymbol{k}_\varphi = \cos\varphi\, \boldsymbol{k}_0 + \sin\varphi\, \boldsymbol{k}_{\frac{\pi}{2}}, \tag{4.24}$$

[2] We will denote generators of screw transformations by χ.

[3] I.e. assume $\boldsymbol{n}_0 \cdot \boldsymbol{n}_1 \neq \pm 1$ for the Euclidean inner product "$\cdot$".

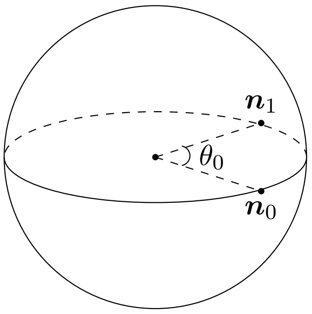
where $\boldsymbol{k}_0$ is perpendicular to the plane spanned by $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$:

$$\boldsymbol{k}_0 = \frac{\boldsymbol{n}_0 \times \boldsymbol{n}_1}{\|\boldsymbol{n}_0 \times \boldsymbol{n}_1\|}, \tag{4.25}$$

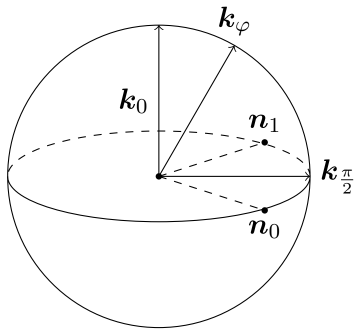
and $\boldsymbol{k}_{\frac{\pi}{2}}$ is in the plane spanned by $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$, as

$$\boldsymbol{k}_{\frac{\pi}{2}} = \frac{\boldsymbol{n}_0 + \boldsymbol{n}_1}{\|\boldsymbol{n}_0 + \boldsymbol{n}_1\|}, \tag{4.26}$$

see also Figure 4.2b. These normalisation constants can also be determined more explicitly, as



(a) Two orientations



(b) Rotation axes

Figure 4.2: For $\boldsymbol{n}_0 \neq \pm\boldsymbol{n}_1$ we can define the axes $\boldsymbol{k}_\varphi$, which allow us to define the generators $\chi(\varphi) = \tilde{\chi}(\boldsymbol{k}_\varphi)$.

if $\theta_0 \in (0, \pi)$ is defined as the angle between $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$, we have that

$$\|\boldsymbol{n}_0 \times \boldsymbol{n}_1\| = \|\boldsymbol{n}_0\|\|\boldsymbol{n}_1\| \sin\theta_0 = \sin\theta_0 > 0, \text{ and}$$
$$\|\boldsymbol{n}_0 + \boldsymbol{n}_1\|^2 = (\boldsymbol{n}_0 + \boldsymbol{n}_1) \cdot (\boldsymbol{n}_0 + \boldsymbol{n}_1) = 2(1 + \cos\theta_0),$$

such that $\|\boldsymbol{n}_0 + \boldsymbol{n}_1\| = 2\cos\frac{\theta_0}{2}$ and

$$\boldsymbol{k}_0 = \frac{\boldsymbol{n}_0 \times \boldsymbol{n}_1}{\sin\theta_0}, \quad \text{and} \quad \boldsymbol{k}_{\frac{\pi}{2}} = \frac{\boldsymbol{n}_0 + \boldsymbol{n}_1}{2\cos\frac{\theta_0}{2}}. \tag{4.27}$$

For later use, we compute the inner product between $\boldsymbol{k}_\varphi$ and either orientation $\boldsymbol{n}_0$ or $\boldsymbol{n}_1$. By symmetry this is the same for both orientations, and given by, for both $i = 0, 1$:

$$\boldsymbol{n}_i \cdot \boldsymbol{k}_\varphi = (\boldsymbol{n}_i \cdot \boldsymbol{k}_0)\cos\varphi + (\boldsymbol{n}_i \cdot \boldsymbol{k}_{\frac{\pi}{2}})\sin\varphi = \cos\frac{\theta_0}{2}\sin\varphi. \tag{4.28}$$

Similarly to θ_0, which is the rotation angle around $\boldsymbol{k}_0$, we can define the rotation angle θ_φ around axis $\boldsymbol{k}_\varphi$ for any arbitrary φ. Using the inner product this is given by

$$\cos\theta_\varphi = \frac{\boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p}{\|\boldsymbol{n}_0^p\|\|\boldsymbol{n}_1^p\|}, \tag{4.29}$$

where $\boldsymbol{n}_0^p$ and $\boldsymbol{n}_1^p$ are the projections of $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ onto the plane of rotation:

$$\boldsymbol{n}_i^p = \boldsymbol{n}_i - (\boldsymbol{k}_\varphi \cdot \boldsymbol{n}_i)\boldsymbol{k}_\varphi, \tag{4.30}$$

for $i = 0, 1$, such that, using Equation (4.28), we can write Eq. (4.29) as

$$\cos\theta_\varphi = \frac{\boldsymbol{n}_0 \cdot \boldsymbol{n}_1 - (\boldsymbol{n}_0 \cdot \boldsymbol{k}_\varphi)(\boldsymbol{n}_1 \cdot \boldsymbol{k}_\varphi)}{\sqrt{1 - (\boldsymbol{n}_0 \cdot \boldsymbol{k}_\varphi)^2}\sqrt{1 - (\boldsymbol{n}_1 \cdot \boldsymbol{k}_\varphi)^2}} = \frac{\boldsymbol{n}_0 \cdot \boldsymbol{n}_1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi}{1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi}. \tag{4.31}$$

As the cosine function is 2π-periodic and even, this does not uniquely determine θ_φ. It only determines $\theta_\varphi \in [0, \pi]$, while both $-\theta_\varphi$ and any added multiple of 2π are also a valid solution. As we want to keep the direction of rotation consistent relative to the axis of rotation, we make the following piecewise definition of θ_φ:

$$\theta_\varphi = \begin{cases} \arccos \frac{\boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p}{\|\boldsymbol{n}_0^p\| \|\boldsymbol{n}_1^p\|} + 2\pi m, & -\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2} \\ 2\pi - \arccos \frac{\boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p}{\|\boldsymbol{n}_0^p\| \|\boldsymbol{n}_1^p\|} + 2\pi m, & \frac{\pi}{2} \leqslant |\varphi| \leqslant \pi \end{cases}, \quad m \in \mathbb{Z}. \tag{4.32}$$

For simplicity however, we fix $m = 0$, i.e. there are no additional rotations.

4.2.2 Position parametrisation of the screw motion

Now that the rotation of the orientations has been determined, we can define the transformation for the positions, or in general from $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0)$ to $\boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1)$ in terms of $\boldsymbol{c}, \boldsymbol{t}$ and $\boldsymbol{\omega}$. We know that a screw transformation maps as in Eq. (4.21), so

$$(\boldsymbol{x}, \boldsymbol{n}) \mapsto (\boldsymbol{c} + \boldsymbol{R}(\boldsymbol{x} - \boldsymbol{c}) + \boldsymbol{t}, \boldsymbol{R}\boldsymbol{n}), \tag{4.33}$$

with $\boldsymbol{R}\boldsymbol{t} = \boldsymbol{t}$. From the above discussion we know that $\boldsymbol{R} = e^{\boldsymbol{M}_\omega}$, where $\boldsymbol{M}_\omega$ corresponds to the angular velocity vector

$$\boldsymbol{\omega}(\varphi) = \theta_\varphi \boldsymbol{k}_\varphi \tag{4.34}$$

as defined above. Hence, $\boldsymbol{R}\boldsymbol{t} = \boldsymbol{t}$ implies that $\boldsymbol{\omega}(\varphi)$ and $\boldsymbol{t} = \boldsymbol{t}(\varphi)$ are parallel. Defining $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_0$ the vector from $\boldsymbol{x}_0$ to $\boldsymbol{x}_1$, we can decompose this translation vector into a translation along the axis of rotation and in the plane perpendicular to the axis of rotation, see also in Figure 4.3a, as

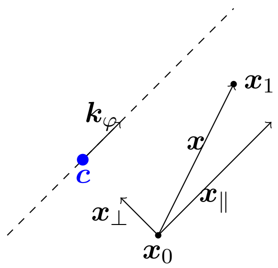
$$\boldsymbol{x}_\parallel = (\boldsymbol{x} \cdot \boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi, \tag{4.35a}$$

$$\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel = \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi. \tag{4.35b}$$

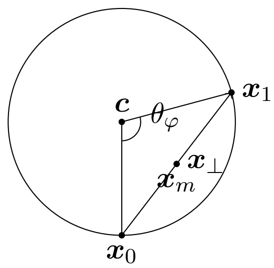
Then $\boldsymbol{t}(\varphi)$ is the translation parallel to $\boldsymbol{\omega}(\varphi)$, such that

$$\boldsymbol{t}(\varphi) = \boldsymbol{x}_\parallel = ((\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot \boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi \tag{4.36}$$

This means that the sole remaining objective is to determine the point $\boldsymbol{c}$ as a function of φ anywhere on the axis of rotation. For this, we look directly at the plane of rotation, see Figure 4.3b (i.e. $\boldsymbol{t}(\varphi)$ lies perpendicular to the paper). In this diagram we see the midpoint



(a) Decomposition of translation into vectors parallel and perpendicular to the axis of rotation $\boldsymbol{k}_\varphi$, pictured by the dashed line.



(b) Trigonometric diagram to determine any point c on the axis of rotation (perpendicular to the image).

Figure 4.3

$\boldsymbol{x}_m = \frac{\boldsymbol{x}_0+\boldsymbol{x}_1}{2}$ of $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$. Using trigonometry, we can see that the distance between $\boldsymbol{c}$ and $\boldsymbol{x}_m$ is $\|\boldsymbol{x}_\perp\|/(2\tan\frac{\theta_\varphi}{2})$, and from $\boldsymbol{x}_m$ the connecting line segment is in the direction $\boldsymbol{k}_\varphi \times \boldsymbol{x}_\perp$, such that

$$\boldsymbol{c} = \boldsymbol{x}_m + \cot\frac{\theta_\varphi}{2}\boldsymbol{k}_\varphi \times \frac{\boldsymbol{x}_\perp}{2} = \boldsymbol{x}_m + \cot\frac{\theta_\varphi}{2}\boldsymbol{k}_\varphi \times \frac{\boldsymbol{x}}{2}, \tag{4.37}$$

where the last step follows from applying Eq. (4.35b). Making the φ-dependence more explicit, this can be expressed as

$$\boldsymbol{c} = \boldsymbol{c}(\varphi) = \boldsymbol{x}_m + \frac{1}{4}\cot\frac{\theta_0}{2}((1+\cos 2\varphi)(\boldsymbol{k}_0 \times \boldsymbol{x}) + \sin 2\varphi(\boldsymbol{k}_{\frac{\pi}{2}} \times \boldsymbol{x})). \tag{4.38}$$

From this it can be read that the curve $\boldsymbol{c}(\varphi)$ is π-periodic and forms an ellipse with centre $\boldsymbol{x}_m + \frac{1}{4}\cot\frac{\theta_0}{2}\boldsymbol{k}_0 \times \boldsymbol{x}$ and with axes $\frac{1}{4}\cot\frac{\theta_0}{2}\boldsymbol{k}_0 \times \boldsymbol{x}$ and $\frac{1}{4}\cot\frac{\theta_0}{2}\boldsymbol{k}_{\frac{\pi}{2}} \times \boldsymbol{x}$.

Remark. *The curve $\boldsymbol{c}(\varphi)$ has a different period than the axis $\boldsymbol{k}_\varphi$, such that, when plotting them together, a Möbius strip is formed, see Fig. 4.4.*

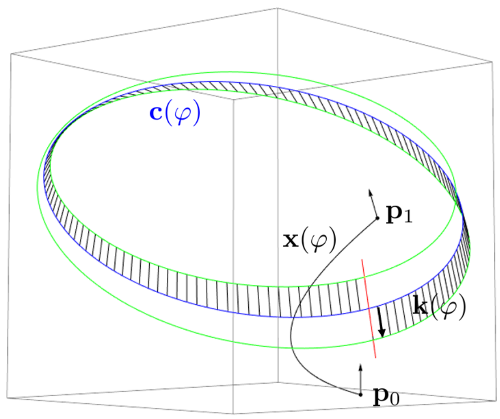


Figure 4.4: The curve $\boldsymbol{c}(\varphi)$ (blue) and axes $\boldsymbol{k}_\varphi$ (black, plotted every $\frac{\pi}{128}$) anchored at $\boldsymbol{c}(\varphi)$ as a function of $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, showing that this plots a Möbius strip. The green lines visualise the outer edges of the axes $\pm\boldsymbol{k}_\varphi$. Also pictured are the relevant position-orientations $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$, as well as the (spatial) screw curve $\boldsymbol{x}(\varphi)$, rotating around the red axis ($\varphi = \frac{\pi}{2}$).

4.2.3 Parametrisation of generators and their norm

Hence, we have determined all parameters of the generator as given in Equation (4.23). As these all depend on the parameter φ through the axis $\boldsymbol{k}_\varphi$, we can write:

$$\boxed{\begin{array}{l}\chi(\varphi) = \tilde{\chi}(\boldsymbol{k}_\varphi) \text{ with} \\ \tilde{\chi}(\boldsymbol{k}) = \left(\theta\left(\boldsymbol{x}_m \times \boldsymbol{k} + \frac{1}{2}\cot\frac{\theta}{2}((\boldsymbol{k}\cdot\boldsymbol{x})\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{k})\right) + (\boldsymbol{x}\cdot\boldsymbol{k})\boldsymbol{k}, \theta \boldsymbol{M}_{\boldsymbol{k}}\right),\end{array}} \tag{4.39}$$

where $\boldsymbol{x}_m = \frac{\boldsymbol{x}_0+\boldsymbol{x}_1}{2}, \boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_0$ and $\theta = \theta(\boldsymbol{k})$, implicitly given by (for projected orientations $\boldsymbol{n}_i^p$ given similarly as in Eq. (4.30))

$$\cos\theta = \frac{\boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p}{\|\boldsymbol{n}_0^p\|\|\boldsymbol{n}_1^p\|} = \frac{\boldsymbol{n}_0\cdot\boldsymbol{n}_1 - (\boldsymbol{n}_0\cdot\boldsymbol{k})(\boldsymbol{n}_1\cdot\boldsymbol{k})}{\sqrt{1-(\boldsymbol{n}_0\cdot\boldsymbol{k})^2}\sqrt{1-(\boldsymbol{n}_1\cdot\boldsymbol{k})^2}}. \tag{4.40}$$

Remark. *Explicitly this means that θ can be given by the 2-argument arctangent function as*

$$\theta = \operatorname{arctan2}(\boldsymbol{k} \cdot (\boldsymbol{n}_0^p \times \boldsymbol{n}_1^p), \boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p) = \arg(\boldsymbol{n}_0^p \cdot \boldsymbol{n}_1^p + i\boldsymbol{k} \cdot (\boldsymbol{n}_0^p \times \boldsymbol{n}_1^p)) \in [0, 2\pi). \tag{4.41}$$

The next step now is to compute the norm of this generator to compute the logarithmic distance as defined in Eq. (2.21). As the construction above is on the manifold itself however, we have chosen to use the logarithmic distance on the manifold instead of on the Lie group, Definition 2.26:

$$\rho_{\mathcal{G}}(\boldsymbol{p}_0, \boldsymbol{p}_1) = \inf_{\varphi \in [-\pi, \pi)} L_{\mathcal{G}}(\eta_{\chi(\varphi)} \triangleright \boldsymbol{p}_0) = \inf_{\varphi \in [-\pi, \pi)} \|\chi(\varphi) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}, \tag{4.42}$$

noting that this is equivalent via Lemma 2.3 and Corollary 4.2 below. The only difference is in the space on which to place the (pseudo)metric; $\mathcal{G}$ on $\mathbb{M}_3$ or $\widetilde{\mathcal{G}}$ on G. Discussion of the forms of these corresponding metrics will be given in Section 4.3.1. Here we use the SE(3)-invariant metric $\mathcal{G}$ on $\mathbb{M}_3$, as given in Equation (3.18).

With this parametrisation, we can write the computation of this logarithmic distance as an optimisation problem over φ, as all exponential curves mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$ are now written as a function of φ. This has the added benefit that, while for simplicity we fix $\boldsymbol{x}_0 = \boldsymbol{0}$, the reference orientation $\boldsymbol{n}_0$ may be chosen arbitrarily. In order to properly set up the optimisation problem, we define $(\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}) \in T_{\boldsymbol{p}_0}\mathbb{M}_3$ as the action of the generator $\chi(\varphi)$ on $\boldsymbol{p}_0 = (\boldsymbol{x}_0, \boldsymbol{n}_0) \in \mathbb{M}_3$, i.e.

$$\boxed{\begin{array}{l}(\dot{\boldsymbol{x}}(\varphi), \dot{\boldsymbol{n}}(\varphi)) := \chi(\varphi) \triangleright (\boldsymbol{x}_0, \boldsymbol{n}_0) = (\boldsymbol{M}_{\omega(\varphi)}(\boldsymbol{x}_0 - \boldsymbol{c}(\varphi)) + \boldsymbol{t}(\varphi), \boldsymbol{M}_{\omega(\varphi)}\boldsymbol{n}_0),\\ \text{with } \boldsymbol{\omega}(\varphi) \text{ given by Eq. (4.34), with } \boldsymbol{t}(\varphi) \text{ given by Eq. (4.36),}\\ \text{and with } \boldsymbol{c}(\varphi) \text{ given by Eq. (4.37) or (4.38).}\end{array}} \tag{4.43}$$

We define the square of the objective function in Equation (4.42) as the function E:

$$E(\varphi) := \|\chi(\varphi) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}^2 = \mathcal{G}_{\boldsymbol{p}_0}(\chi(\varphi) \triangleright \boldsymbol{p}_0, \chi(\varphi) \triangleright \boldsymbol{p}_0), \quad \text{so} \quad \rho_{\mathcal{G}}(\boldsymbol{p}_0, \boldsymbol{p}_1)^2 = \inf_{\varphi \in [-\pi, \pi)} E(\varphi). \tag{4.44}$$

Using the metric $\mathcal{G}$ as defined in Equation (3.18), this function can be expressed as

$$\boxed{\begin{array}{l}E(\varphi) = \|(\dot{\boldsymbol{x}}(\varphi), \dot{\boldsymbol{n}}(\varphi))\|_{\mathcal{G}|_{\boldsymbol{p}_0}}^2 = w_1|\dot{\boldsymbol{x}}(\varphi) \cdot \boldsymbol{n}_0|^2 + w_2\|\dot{\boldsymbol{x}}(\varphi) \times \boldsymbol{n}_0\|^2 + w_3\|\dot{\boldsymbol{n}}(\varphi)\|^2,\\ \text{with } (\dot{\boldsymbol{x}}(\varphi), \dot{\boldsymbol{n}}(\varphi)) \text{ given by Eq. (4.43),}\end{array}} \tag{4.45}$$

which is the form we will (analytically) optimise in this report.

Using Lemma 2.3, we have the following corollary about the interpretation of $E(\varphi)$ as defined in Eq. (4.44).

Corollary 4.2. *For any metric $\mathcal{G}$ on $\mathbb{M}_3$ and the pseudometric $\widetilde{\mathcal{G}}$ on G defined from $\mathcal{G}$ via Lemma 2.2, we have the following chain of equalities*

$$E(\varphi) := \|\chi(\varphi) \triangleright \boldsymbol{p}_0\|_{\mathcal{G}}^2 = \|\chi(\varphi)\|_{\widetilde{\mathcal{G}}}^2 = \|\log e^{\chi(\varphi)}\|_{\widetilde{\mathcal{G}}}^2 = \rho_{\widetilde{\mathcal{G}}}(e^{\chi(\varphi)}, e)^2, \tag{4.46}$$

where $\rho_{\widetilde{\mathcal{G}}}$ is defined as in Equation (2.21).

Remark. *This means that while we have defined $E(\varphi)$ as the squared norm of the action of a generator $\chi(\varphi)$ on reference $\boldsymbol{p}_0$, it can also be interpreted as the squared norm of the generator itself. This is shown to be equal to the squared logarithmic distance from the rototranslation generated by $\chi(\varphi)$ to identity e. As the generators $\varphi \mapsto \chi(\varphi)$ have been constructed such that all map $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$, these form the fibre corresponding to $\boldsymbol{p}_1$ and minimising E minimises the logarithmic norm over the parametrised fibre.*

Special cases of interest

Upon fixing $\boldsymbol{x}_0 = \boldsymbol{0}$, we see that $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{n}}$ from Equation (4.43) can be simplified as

$$\dot{\boldsymbol{x}}(\varphi) = \boldsymbol{t}(\varphi) + \boldsymbol{c}(\varphi) \times \boldsymbol{\omega}(\varphi), \tag{4.47a}$$

$$\dot{\boldsymbol{n}}(\varphi) = \theta_\varphi \left(\cos\varphi \frac{(\boldsymbol{n}_0 \times \boldsymbol{n}_1) \times \boldsymbol{n}_0}{\sin\theta_0} + \sin\varphi \frac{\boldsymbol{n}_1 \times \boldsymbol{n}_0}{2\cos\frac{\theta_0}{2}} \right). \tag{4.47b}$$

In the definition of coplanarity (Def. 3.2), Equation (3.2) immediately reduces to

$$\boldsymbol{p}_0, \boldsymbol{p}_1 \text{ are coplanar } \Leftrightarrow \boldsymbol{x}_1 \cdot \boldsymbol{k}_0 = 0. \tag{4.48}$$

Similarly, we could define the condition $\boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}} = 0$ to be called "in mirror".

$$\boldsymbol{p}_0, \boldsymbol{p}_1 \text{ are 'in mirror' } \Leftrightarrow \boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}} = 0. \tag{4.49}$$

For an intuitive visualisation of these two special cases of interest see Figure 3.1.

4.3 From logarithmic to screw coordinates

In this next section we discuss the relation between logarithmic coordinates and the coordinates arising from the screw transformation. It can be seen that a screw transformation can be written as a regular rototranslation combined with a conjugation with $(\boldsymbol{c}, \boldsymbol{I}) \in G$, as this transforms the rotation about the origin into a rotation about $\boldsymbol{c} \in \mathbb{R}^3$ on the screw axis. Hence, this can be written as

$$\exp\left(\boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)}\right) = \exp\left(\mathrm{Ad}_{(\boldsymbol{c},\boldsymbol{I})}(\boldsymbol{d}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{d}^{(2)} \cdot \boldsymbol{A}_{(2)})\right), \tag{4.50}$$

where $\boldsymbol{c}^{(1)}$ is the total translation $\boldsymbol{x}$, $\boldsymbol{d}^{(2)} = \boldsymbol{\omega} = \boldsymbol{c}^{(2)}$ is the total angular velocity vector, unchanged under conjugation, and $\boldsymbol{d}^{(1)} = \boldsymbol{t} = \frac{\boldsymbol{c}^{(1)} \cdot \boldsymbol{c}^{(2)}}{\boldsymbol{c}^{(2)} \cdot \boldsymbol{c}^{(2)}} \boldsymbol{c}^{(2)}$ is the translation along the axis of rotation. We can use the construction of $\boldsymbol{c}$ in Equation (4.37) to express $\boldsymbol{c}$ in terms of the logarithmic coordinates $\boldsymbol{c}^{(1)}$ and $\boldsymbol{c}^{(2)}$, remembering that $\boldsymbol{c}$ is determined up to an additional factor $\boldsymbol{t}$[4]. This means we can replace $\boldsymbol{x}_m$ by $\frac{1}{2}\boldsymbol{x}_\perp$, and by knowing that $\theta_\varphi = \|\boldsymbol{\omega}_\varphi\| = \|\boldsymbol{c}^{(2)}\|$ and $\boldsymbol{k}_\varphi = \frac{\boldsymbol{c}^{(2)}}{\|\boldsymbol{c}^{(2)}\|}$, this translates to

$$\boldsymbol{c} = \frac{1}{2}\left(\boldsymbol{x}_\perp + \cot\left(\frac{\|\boldsymbol{c}^{(2)}\|}{2}\right) \frac{\boldsymbol{c}^{(2)}}{\|\boldsymbol{c}^{(2)}\|} \times \boldsymbol{x}_\perp \right), \tag{4.51}$$

with $\boldsymbol{x}_\perp = \boldsymbol{c}^{(1)} - \boldsymbol{d}^{(1)}$.

4.3.1 Log- and screw coordinates and the norm

In logarithmic coordinates for $g = \exp(\boldsymbol{c}^{(1)} \cdot \boldsymbol{A}_{(1)} + \boldsymbol{c}^{(2)} \cdot \boldsymbol{A}_{(2)})$, the norm on the tangent space induced by the inner product $\widetilde{\mathcal{G}}$ is given by, using the definition in [16]:

$$\|\log g\|^2_{\widetilde{\mathcal{G}}} = \sum_{i=1}^{6} g_{ii}|c^i|^2 \tag{4.52a}$$

$$= g_{11}\left(|c^1|^2 + |c^2|^2\right) + g_{33}|c^3|^2 + g_{44}\left(|c^4|^2 + |c^5|^2\right) + g_{66}|c^6|^2 \tag{4.52b}$$

$$= g_{11}\|\boldsymbol{c}^{(1)} \times \boldsymbol{a}\|^2 + g_{33}|\boldsymbol{c}^{(1)} \cdot \boldsymbol{a}|^2 + g_{44}\|\boldsymbol{c}^{(2)} \times \boldsymbol{a}\|^2 + g_{66}|\boldsymbol{c}^{(2)} \cdot \boldsymbol{a}|^2 \tag{4.52c}$$

[4] Adding any multiple of $\boldsymbol{t}$ to $\boldsymbol{c}$ still results in a point on the axis of rotation.

for reference axis $\boldsymbol{a} = \boldsymbol{n}_0 = \boldsymbol{e}_z$.

Using the relation between logarithmic and screw coordinates as discussed in the previous section, Equation (4.43) can be rephrased as

$$(\dot{\boldsymbol{x}}, \dot{\boldsymbol{n}}) = (\boldsymbol{d}^{(2)} \times (\boldsymbol{x}_0 - \boldsymbol{c}) + \boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)} \times \boldsymbol{n}_0) \tag{4.53}$$

and as we calculated before, see Eq. (4.23), the logarithm of such a screw transformation is

$$\log g = (\boldsymbol{c} \times \boldsymbol{d}^{(2)} + \boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}) = (\boldsymbol{c}^{(1)}, \boldsymbol{c}^{(2)}), \tag{4.54}$$

such that $\boldsymbol{c}^{(1)} = \dot{\boldsymbol{x}}$. Hence, upon comparing the coefficients w_i in Equation (4.45) with the coefficients g_{ii} in Equation (4.52c), and using that these should be equal if $\widetilde{\mathcal{G}}$ is induced by $\mathcal{G}$ (Lemma 2.3), we see that $g_{11} = w_2, g_{33} = w_1$ and $g_{44} = w_3$.

Remark. *Notice that there is no correspondence between any coefficient w_i and g_{66}. This is due to the fact that for an induced metric we have that $g_{66} = 0$, leading to the degeneracy along H and why we call it a pseudometric. As it has been shown that the value of g_{66} is irrelevant for the computation of geodesics [16], such a correspondence is irrelevant.*

4.3.2 The simple section $\sigma_{\mathbf{MAV}}$ in screw coordinates

In logarithmic coordinates, the simple section σ_{MAV} is given by the condition $c^6 = 0$, recall Section 4.1, which is equivalent to $\boldsymbol{\omega} \cdot \boldsymbol{n}_0 = 0$ for the fixed axis $\boldsymbol{n}_0 = \boldsymbol{e}_z$. In screw coordinates, for any general starting orientation $\boldsymbol{n}_0$ this can also be expressed as $\varphi = 0$, such that the simple section has an equally simple description in screw coordinates. Then also the MAV generator is given by $\chi(0)$ and the MAV section by $\sigma_{\text{MAV}} = \exp(\chi(0))$.

The section σ_ρ on the other hand, recall Eq. (3.23), corresponds to the value of φ that minimises the function E in Equation (4.45). Hence these sections coincide when

$$\underset{\varphi \in [-\pi, \pi)}{\arg\min}\, E(\varphi) = 0, \tag{4.55}$$

which is the problem of interest for the remainder of the report.

4.4 Analysing the norm

In order to study when Equation (4.55) is satisfied and when violated, we rewrite the function E as

$$E(\varphi) = (w_1 - w_2)|\dot{\boldsymbol{x}} \cdot \boldsymbol{n}_0|^2 + w_2\|\dot{\boldsymbol{x}}\|^2 + w_3\|\dot{\boldsymbol{n}}\|^2, \text{ with } \dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(\varphi),\ \dot{\boldsymbol{n}} = \dot{\boldsymbol{n}}(\varphi) \tag{4.56}$$

With this we first investigate the evenness of this function (in Lemma 4.2), after which we compute the derivative of E at $\varphi = 0$ (full computations in Appendix A) in order to determine whether $\varphi = 0$ is a minimiser, a maximiser, or possibly neither by the second derivative test.

Recall that a function f is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$.

Lemma 4.2. *All odd terms in $E(\varphi)$ contain factors of the form $(\boldsymbol{x}_1 \cdot \boldsymbol{k}_0)(\boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}})$, i.e. when the position-orientations $\boldsymbol{p}_0, \boldsymbol{p}_1$ are coplanar or in mirror, E is an even function.*

Proof. From Equation (4.32) we see that θ_φ is fully even. With this we see that also $\|\dot{\boldsymbol{n}}\|^2$ is fully even:

$$\|\dot{\boldsymbol{n}}\|^2 = \theta_\varphi^2 \left(1 - \frac{1 + \cos\theta_0}{2} \sin^2\varphi\right).$$

The odd part of $\|\dot{\boldsymbol{x}}(\varphi)\|^2$ can be written as

$$\frac{\|\dot{\boldsymbol{x}}(\varphi)\|^2 - \|\dot{\boldsymbol{x}}(-\varphi)\|^2}{2} = \frac{\sin 2\varphi}{2}\left(2 - \frac{\theta_\varphi^2}{1-\cos\theta_\varphi}\right)(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}),$$

and the odd part of $|\dot{\boldsymbol{x}}(\varphi)\cdot\boldsymbol{n}_0|^2$ can be written as the product

$$\begin{aligned}\frac{|\dot{\boldsymbol{x}}(\varphi)\cdot\boldsymbol{n}_0|^2 - |\dot{\boldsymbol{x}}(-\varphi)\cdot\boldsymbol{n}_0|^2}{2} = & \frac{(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})}{8} \\ & \cdot\left(2\theta_\varphi\frac{\cos\varphi}{\sin\frac{\theta_0}{2}} + \cos\frac{\theta_0}{2}\sin\varphi\left(4\sin\varphi - \cot\frac{\theta_0}{2}\theta_\varphi\sin 2\varphi\right)\right) \\ & \cdot\left(\cos\frac{\theta_0}{2}\cos\varphi\left(4\sin\varphi - \cot\frac{\theta_0}{2}\theta_\varphi\sin 2\varphi\right) - 2\theta_\varphi\sin\varphi\sin\frac{\theta_0}{2}\right).\end{aligned}$$

□

From this follows the following Theorem:

Theorem 4.1 (Coplanar extremum)**.** *In the coplanar (as well as the in mirror) case, see Equation* (4.48)*, we have that the function E as defined in Equation* (4.45) *has its extremum at $\varphi = 0$.*

Remark. *This is assuming that $E(\varphi)$ is not a constant. If that is the case, the problem is trivial. Furthermore, this does not allow us to know whether this extremum is a minimum, in which case $\sigma_{\mathrm{MAV}} = \sigma_\rho$, or a (local) maximum. This is the subject of investigation in the coplanar case.*

Remark. *If this function is even (using Lemma 4.2), it must have an extremum at $\varphi = 0$, proving this Theorem 4.1. This simple observation greatly simplifies the proof of the same result in Lemma 2 of [16], showing the advantage of the screw-motion parametrisation.*

Observe that in the spatially isotropic case, i.e. $w_1 = w_2 = w$ and $\zeta = 1$, see the discussion in Section 3.3, the first term in Equation (4.56) vanishes. This means that we can state the following Theorem.

Theorem 4.2 (Derivatives in the Spatially Isotropic Metric).
Consider the auxiliary function E as a function of $\varphi \in [-\pi, \pi)$, as given in Equation (4.56)*. In the spatially isotropic case, i.e. $w_1 = w_2 = w$, we have that its first and second derivative at $\varphi = 0$ are given by*

$$\boxed{E'(0) = w(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})\left(2 - \frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}}\right),} \tag{4.57}$$

and

$$\boxed{\begin{aligned} E''(0) &= w\left(\frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}} - 2\right)\left(|\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2 - \left|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right|^2\right) \\ &\quad + \theta_0 w\cot\frac{\theta_0}{2}\left(1 - \frac{\theta_0}{2}\cot\frac{\theta_0}{2}\right)\left(\|\boldsymbol{x}_1\|^2 - |\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2\right) \\ &\quad + 2w_3\theta_0\left(\sin\theta_0 - \theta_0\cos^2\frac{\theta_0}{2}\right).\end{aligned}} \tag{4.58}$$

Proof. See the computations in Sections A.1 and A.2 of the Appendix. □

More generally, in the spatially anisotropic metric these derivatives are given by the following Theorem.

Theorem 4.3 (Derivatives in the general Spatially Anisotropic Metric case). *Consider the auxiliary function E as a function of $\varphi \in [-\pi, \pi)$, as given in Equation (4.56). In the spatially anisotropic case, i.e. $w_1 \neq w_2$, we have that its first and second derivative are given by*

$$\boxed{E'(0) = (\boldsymbol{x}_1 \cdot \boldsymbol{k}_0)(\boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}}) \left(w_2 \left(2 - \frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}} \right) + (w_1 - w_2) \cot \frac{\theta_0}{2} \frac{\sin\theta_0 - \theta_0}{\operatorname{sinc}\theta_0} \right),} \tag{4.59}$$

where $\operatorname{sinc} z = \frac{\sin z}{z}$, *and*

$$\boxed{\begin{aligned} E''(0) &= w_2 \left(\frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}} - 2 \right) \left(|\boldsymbol{x}_1 \cdot \boldsymbol{k}_0|^2 - \left| \boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}} \right|^2 \right) \\ &\quad + \theta_0 w_2 \cot\frac{\theta_0}{2} \left(1 - \frac{\theta_0}{2} \cot \frac{\theta_0}{2} \right) \left(\|\boldsymbol{x}_1\|^2 - |\boldsymbol{x}_1 \cdot \boldsymbol{k}_0|^2 \right) + 2w_3\theta_0 \left(\sin\theta_0 - \theta_0 \cos^2\frac{\theta_0}{2} \right) \\ &\quad + (w_1 - w_2) \left(\theta_0 \cot\frac{\theta_0}{2} \left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0 \cot\frac{\theta_0}{2} \right) \left| \boldsymbol{x}_1 \cdot \boldsymbol{k}_{\frac{\pi}{2}} \right|^2 + |\boldsymbol{x}_1 \cdot \boldsymbol{k}_0|^2 \frac{|\sin\theta_0 - \theta_0|^2}{2\sin^2\frac{\theta_0}{2}} \right). \end{aligned}} \tag{4.60}$$

Proof. These derivatives are given by those in the previous Theorem, combined with the derivatives of $|\dot{x} \cdot \dot{n}|^2$ as computed in Section A.3 in the appendix. $\square$

In the remainder of this chapter, we discuss to which rototranslation the section σ_{MAV} corresponds when the orientations $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are colinear, as in that case the parameter φ is ill-defined. In the following chapter we continue the analysis of the derivatives of $E(\varphi)$ as given in the previous Theorems, both in the coplanar and non-coplanar case.

4.5 Colinear orientations

The previous discussion of the screw transformation only works under the condition that $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are not colinear, so $\boldsymbol{n}_0 \neq \pm\boldsymbol{n}_1$[5], as then the axis $\boldsymbol{k}_0$ (and also possibly $\boldsymbol{k}_{\frac{\pi}{2}}$) is undefined. As at least one of these basis elements is undefined, the linear combination $\boldsymbol{k}_\varphi$ (see Eq. (4.24)) and the parameter φ are also undefined. This is a minor short-coming of this parametrisation, and for this reason we expressed the dependence of the generator on the axis $\boldsymbol{k}$ in Eq. (4.39).

In the case that $\boldsymbol{n}_0 = \boldsymbol{n}_1$ we have that the angle $\theta = 0$, such that the generator reduces to $\tilde{\chi} = (\boldsymbol{x}_1, \boldsymbol{0})$, i.e. there is a single unique generator consisting of no rotation and only translation. In the other case, i.e. $\boldsymbol{n}_0 = -\boldsymbol{n}_1$ we have that the angle $\theta = \pi$ independent of the axis $\boldsymbol{k}$, and as all allowable axes lie in the bisector plane of $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$, we have that $\boldsymbol{k} \in \langle \boldsymbol{n}_0 \rangle^\perp$. Hence, the generator reduces to

$$\tilde{\chi}(\boldsymbol{k}) = (\pi(\boldsymbol{x}_1 \times \boldsymbol{k}) + (\boldsymbol{x}_1 \cdot \boldsymbol{k})\boldsymbol{k}, \pi \boldsymbol{M}_{\boldsymbol{k}}). \tag{4.61}$$

Recall $\boldsymbol{x}_0 = \boldsymbol{0}$.

In the general, non-colinear case, we are interested in whether the rototranslation corresponding to the MAV section, i.e. $\boldsymbol{k} \cdot \boldsymbol{n}_0 = 0$, see Sec. 4.3.2, minimises the norm. In this case that question is ill-defined, as either there is only a single rototranslation (that may or may not be σ_{MAV}), or all viable rototranslations satisfy the MAV condition.

[5] Equivalently they are colinear when $|\boldsymbol{n}_0 \cdot \boldsymbol{n}_1| = 1$, or $\theta_0 = 0 \vee \theta_0 = \pi$.

Chapter 5

Critical point analysis of the optimisation function $E(\varphi)$

In this chapter we analyse the derivatives of E given in Theorems 4.2 and 4.3. We start with the situation that the start position-orientation $\boldsymbol{p}_0 = (\boldsymbol{0}, \boldsymbol{n}_0)$ and the end position-orientation $\boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1)$ as used in the previous chapter are coplanar, i.e. we assume that the vectors $\boldsymbol{x}_1, \boldsymbol{n}_0$ and $\boldsymbol{n}_1$ all lie in the same plane, or $\boldsymbol{x}_1 \cdot \boldsymbol{k}_0 = 0$, see Definition 3.2 and Equation (4.48).

Under this assumption we have that the first derivatives as given in Theorems 4.2 and 4.3 reduce to

$$\boxed{E'(0) = 0} \tag{5.1}$$

in all cases, as was already predicted in Theorem 4.1. We are however, interested in whether this extremum is a minimum or a local maximum, recall the discussion in Section 4.3.2. For this we will investigate the sign of the second derivatives given in Theorems 4.2 and 4.3. We start with the spatially isotropic case in Section 5.1.1 leading to Theorem 5.1, and handle the spatially anisotropic case later in Section 5.1.2 with Theorem 5.2.

In this coplanar case we can and will write $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_0 + \mu \boldsymbol{n}_1$ for constants $\lambda, \mu \in \mathbb{R}$, see Equation (3.3). The discussion of how to find these parameters is found in Appendix B. As examples we will consider the particular cases $\lambda = 0, \mu = 0$ and $\mu = \pm\lambda$, before considering the general case of arbitrary constants.

Following the discussion of the coplanar case, in Section 5.2 we discuss the situation that $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are not coplanar.

5.1 The coplanar case: analysing the sign of $E''(0)$

This section is reserved for the critical point analysis of the function E at $\varphi = 0$ in the coplanar case. As from Equation (5.1) it follows that E has an extremum at $\varphi = 0$, this reduces to the analysis of the sign of $E''(0)$ to determine the type of extremum: minimum or maximum. We start with the spatially isotropic case, so $w_1 = w_2 = w$ and later provide the generalisation to the spatially anisotropic case in Section 5.1.2, as well as provide lemmas that hold for both of these cases.

5.1.1 The spatially isotropic coplanar case

In the spatially isotropic case, the second derivative is given in Theorem 4.2, Equation (4.58), which under the coplanarity assumption (Eq. (4.48)) reduces to

$$\begin{aligned}E''(0) &= w\theta_0 \cot\frac{\theta_0}{2}\left(1-\frac{\theta_0}{2}\cot\frac{\theta_0}{2}\right)\|\boldsymbol{x}_1\|^2 + w\left(2-\frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}}\right)\left|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right|^2 \\ &\qquad + 2w_3\theta_0\left(\sin\theta_0 - \theta_0\cos^2\frac{\theta_0}{2}\right) \\ &= \frac{\theta_0}{2}\cot\frac{\theta_0}{2}\left(2-\theta_0\cot\frac{\theta_0}{2}\right)(w\|\boldsymbol{x}_1\|^2 + 2w_3(1-\cos\theta_0)) + 2w\left(1-\frac{1}{\operatorname{sinc}^2\frac{\theta_0}{2}}\right)|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}|^2.\end{aligned} \tag{5.2}$$

We define the auxiliary functions $\Psi(\theta)$ and $A(\theta)$ as

$$\Psi(\theta) = \frac{\theta}{2}\cot\frac{\theta}{2}\left(2-\theta\cot\frac{\theta}{2}\right), \tag{5.3a}$$

$$-A(\theta)\Psi(\theta) = 2\left(1-\frac{1}{\operatorname{sinc}^2\frac{\theta_0}{2}}\right), \tag{5.3b}$$

such that both $\Psi(\theta) > 0$ and $A(\theta) > 0$ for $0 < \theta < \pi$, see also Figure 5.1.

With these functions, we can rewrite Equation (5.2) as

$$E''(0) = \Psi(\theta_0)\left(w(\|\boldsymbol{x}_1\|^2 - A(\theta_0)|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}|^2) + 2w_3(1-\cos\theta_0)\right). \tag{5.4}$$

Finally, if we define α_0 to be the angle between $\boldsymbol{x}_1$ and $\boldsymbol{k}_{\frac{\pi}{2}}$, i.e.

$$|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}|^2 = \|\boldsymbol{x}_1\|^2\cos^2\alpha_0, \tag{5.5}$$

we can rewrite Equation (5.4) as

$$\begin{aligned}E''(0) &= \Psi(\theta_0)\left(w\|\boldsymbol{x}_1\|^2(1-A(\theta_0)\cos^2\alpha_0) + 2w_3(1-\cos\theta_0)\right) \\ &\geqslant w\|\boldsymbol{x}_1\|^2\Psi(\theta_0)(1-A(\theta_0)\cos^2\alpha_0),\end{aligned}$$

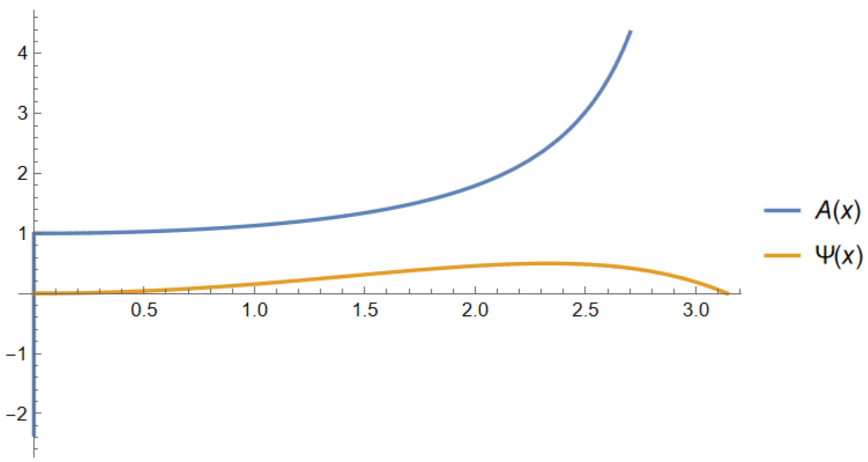


Figure 5.1: Plot of the functions A and Ψ

since the w_3 term is always non-negative. Thus, the entire second derivative is positive if, with equivalence when $w_3 = 0$,

$$\cos^2 \alpha_0 \leqslant \frac{1}{A(\theta_0)} = 1 - \frac{7}{60}\theta_0^2 + \frac{37}{25200}\theta_0^4 + O\left(\theta_0^6\right). \tag{5.6}$$

We will now look at a few simple examples for $\boldsymbol{x}_1$ in terms of $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$.

Example 5.1.1. *We start with $\boldsymbol{x}_1$ in the direction of $\boldsymbol{n}_0$: $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_0$ for some $\lambda \in \mathbb{R}$. From numerical experiments it is known that in this case $\varphi = 0$ is always a minimum. In this case we have that $\alpha_0 = \frac{\theta_0}{2}$, as can be seen either from Equation (5.5) or geometrically, as the vector $\boldsymbol{n}_0 + \boldsymbol{n}_1$ bisects the angle between $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ due to symmetry. This also means that this is equivalent with taking $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_1$.*

So following the condition of Eq. (5.6), we want to check whether

$$\cos^2 \frac{\theta_0}{2} \leqslant \frac{1}{A(\theta_0)}, \quad \textit{or as series:}$$
$$1 - \frac{1}{4}\theta_0^2 + \frac{1}{48}\theta_0^4 + O\left(\theta_0^6\right) \leqslant 1 - \frac{7}{60}\theta_0^2 + \frac{37}{25200}\theta_0^4 + O\left(\theta_0^6\right).$$

This condition is satisfied for all $0 < \theta_0 < \pi$, confirming the numerical experiments.

Next, we take the example that $\boldsymbol{x}_1$ is in the direction of $\boldsymbol{n}_0 + \boldsymbol{n}_1$: $\boldsymbol{x}_1 = \lambda(\boldsymbol{n}_0 + \boldsymbol{n}_1)$. In that case, trivially $\alpha_0 = 0$, such that $\cos^2 \alpha_0 = 1$ and Equation (5.6) is only ever satisfied when $\theta_0 = 0$. Due to the assumption that $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are not colinear, this condition is never satisfied.

Finally, we take the simple example $\boldsymbol{x}_1 = \lambda(\boldsymbol{n}_0 - \boldsymbol{n}_1)$. As this is perpendicular to $\boldsymbol{n}_0 + \boldsymbol{n}_1$, we have that $\cos \alpha_0 = 0$ and Equation (5.6) is satisfied for all $0 < \theta_0 < \pi$.

Now we will investigate the most general case, i.e. $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_0 + \mu \boldsymbol{n}_1$ for arbitrary constants $\lambda, \mu \in \mathbb{R}$. Noting that in that case

$$\cos^2 \alpha_0 = \frac{(\lambda + \mu)^2 \cos^2 \frac{\theta_0}{2}}{\lambda^2 + \mu^2 + 2\lambda\mu \cos \theta_0}, \tag{5.7}$$

we summarise all results in the following Theorem:

Theorem 5.1 (Coplanar solutions to the minimisation in case of spatial isotropy). *Consider the general optimization problem, Eq. (4.55), with $\boldsymbol{x}_0 = \boldsymbol{0}$ for the coplanar case (i.e. Eq. (4.48) holds) in the spatially isotropic setting, i.e. $w_1 = w_2 = w$.*

Then the maximal subset S of end-points $\boldsymbol{p}_1 \in \mathbb{M}_3$ such that

$$\sigma_\rho|_S = \sigma_{\mathrm{MAV}}|_S \quad \Leftrightarrow \quad \varphi = 0 \textit{ minimises } E(\varphi),$$

is given by

$$\boxed{\begin{aligned} S &= \left\{ \boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1) \in \mathbb{M}_3 \;\middle|\; \boldsymbol{x}_1 = \lambda \boldsymbol{n}_0 + \mu \boldsymbol{n}_1, \boldsymbol{n}_0 \cdot \boldsymbol{n}_1 = \cos\theta_0, \widetilde{F}(\lambda, \mu, \theta_0) > 0 \right\}, \\ \widetilde{F}(\lambda, \mu, \theta_0) &= 2w_3(1 - \cos\theta_0) + w\left((\lambda+\mu)^2\left(1 - \cos^2 \tfrac{\theta_0}{2} A(\theta_0)\right) - 4\lambda\mu \sin^2 \tfrac{\theta_0}{2}\right), \end{aligned}} \tag{5.8}$$

where the function A is given by Eq. (5.3b).

Remark. *The set S is maximal, meaning that outside (the closure of) this set S the function $\varphi \mapsto E(\varphi)$ has its maximum at $\varphi = 0$ instead of its minimum.*

Remark. *As the w_3-term is always non-negative, this set expands as w_3 increases, as will be discussed in Lemma 5.3. We define S_0 as the set S for $w_3 = 0$, so $S_0 \subseteq S$ and*

$$S_0 = \left\{ \boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1) \;\middle|\; (\lambda+\mu)^2 \left(1 - \cos^2\frac{\theta_0}{2} A(\theta_0)\right) > 4\lambda\mu \sin^2\frac{\theta_0}{2} \right\}. \tag{5.9}$$

Corollary 5.1. *The inequality defining the set S_0 in Equation (5.9), following from Equation (5.6), can be re-expressed as the following inequality and its Taylor expansion*

$$\frac{(\lambda+\mu)^2 \cos^2\frac{\theta_0}{2}}{\lambda^2+\mu^2+2\lambda\mu\cos\theta_0} \leqslant \frac{1}{A(\theta_0)}, \tag{5.10a}$$

$$1 - \frac{1}{4}\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2 \theta_0^2 + \frac{(\lambda-\mu)^2(\lambda^2-10\lambda\mu+\mu^2)}{48(\lambda+\mu)^4}\theta_0^4 + O(\theta_0^6) \leqslant 1 - \frac{7}{60}\theta_0^2 + \frac{37}{25200}\theta_0^4 + O(\theta_0^6).$$

Remark. *From the previous Corollary 5.1, specifically Eq. (5.10a), can be read that in the cases $\lambda = 0, \mu = 0$ or $\lambda + \mu = 0$, we have that the inequality is satisfied and $\boldsymbol{x}_1 \in S_0 \subseteq S$, irregardless of the choice of $\boldsymbol{n}_1$. For $\boldsymbol{x}_1$ in the direction of $\boldsymbol{n}_0 + \boldsymbol{n}_1$ however, i.e. $\lambda = \mu$, then $\boldsymbol{x}_1 \notin S_0$, although there exists a coefficient w_3 such that $\boldsymbol{x}_1 \in S$. This w_3 is expressed by the inequality*

$$\frac{w_3}{w} > \lambda^2 \cot^2\frac{\theta_0}{2}\left(A(\theta_0) - 1\right). \tag{5.11}$$

Remark. *We provide a term-by-term analysis of the inequality of Corollary 5.1:*

$$\begin{aligned}
&\theta_0^2\text{-}\textit{term:} \qquad \left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2 \geqslant \frac{7}{15} \\
&\qquad \Longrightarrow \lambda \leqslant \frac{11-\sqrt{105}}{4}\mu \vee \lambda \geqslant \frac{11+\sqrt{105}}{4}\mu, \\
&\theta_0^4\text{-}\textit{term:} \qquad \left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2 \frac{\lambda^2-10\lambda\mu+\mu^2}{(\lambda+\mu)^2} \leqslant \frac{37}{525} \\
&\Longrightarrow \frac{403+15\sqrt{371}-\sqrt{30\left(403\sqrt{371}+7700\right)}}{122}\mu \leqslant \lambda \leqslant \frac{403+15\sqrt{371}+\sqrt{30\left(403\sqrt{371}+7700\right)}}{122}\mu.
\end{aligned}$$

Numerically these conditions reduce to:

$$\begin{aligned}
&\theta_0^2\text{-}\textit{term:} \qquad \lambda \leqslant 0.19\mu \vee \lambda \geqslant 5.31\mu, \\
&\theta_0^4\text{-}\textit{term:} \qquad 0.09\mu \leqslant \lambda \leqslant 11.3\mu.
\end{aligned}$$

5.1.2 The general spatially anisotropic coplanar case

In the spatially anisotropic metric, the second derivative of E at $\varphi = 0$ is given in Theorem 4.3, Equation (4.60), which under the coplanarity assumption reduces to

$$E''(0) = \Psi(\theta_0)\left(w_2\|\boldsymbol{x}_1\|^2(1 - A(\theta_0)\cos^2\alpha_0) + 2w_3(1-\cos\theta_0)\right) \tag{5.12a}$$

$$+ (w_1 - w_2)\frac{2\theta_0}{\sin\theta_0}\left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0\cot\frac{\theta_0}{2}\right)\|\boldsymbol{x}_1\|^2\cos^2\frac{\theta_0}{2}\cos^2\alpha_0, \tag{5.12b}$$

where we again define α_0 as the angle between $\boldsymbol{x}_1$ and $\boldsymbol{k}_{\frac{\pi}{2}}$ as in the previous case, Eq. (5.5).

This has a series representation, in the variable θ_0, given by

$$E''(0) = w_2\|\boldsymbol{x}_1\|^2\sin^2\alpha_0\frac{\theta_0^2}{6} + \frac{60w_3 - 3(4w_1-3w_2)\|\boldsymbol{x}_1\|^2\cos^2\alpha_0 - 4w_2\|\boldsymbol{x}_1\|^2}{360}\theta_0^4 + O\left(\theta_0^6\right). \tag{5.13}$$

We will again investigate a few examples.

Example 5.1.2. *Suppose that* $\boldsymbol{x}_1$ *is in the direction of* $\boldsymbol{n}_0$*, so* $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_0$ *for some constant* λ*. Due to symmetry*[1] *the calculation will be identical for* $\boldsymbol{x}_1 = \lambda \boldsymbol{n}_1$*, and in that case we know that* $\alpha_0 = \frac{\theta_0}{2}$ *and* $E''(0)$ *can be written as*

$$\begin{aligned}
E''(0) &= \Psi(\theta_0)\left(w_2\lambda^2\left(1 - A(\theta_0)\cos^2\frac{\theta_0}{2}\right) + 2w_3(1-\cos\theta_0)\right) \\
&\quad + (w_1 - w_2)\lambda^2 \frac{2\theta_0}{\sin\theta_0}\left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0\cot\frac{\theta_0}{2}\right)\cos^4\frac{\theta_0}{2} \\
&= \frac{\lambda^2(5w_2 - 3w_1) + 15w_3}{90}\theta_0^4 + \frac{\lambda^2\left(54w_1 - 70w_2\right) - 189w_3}{7560}\theta_0^6 + O\left(\theta_0^8\right).
\end{aligned}$$

Recall that we only use this metric under the (Riemannian) restriction that $1 \leqslant \zeta \leqslant 4$*, see Section 3.3. Under this restriction, we have that, as in the isotropic example,* $E''(0)$ *is positive for all* $\theta_0 \in (0, \pi)$*.*

Next, let us again consider the case that $\boldsymbol{x}_1 = \lambda(\boldsymbol{n}_0 + \boldsymbol{n}_1)$*, such that* $\alpha_0 = 0$*. In this case we have that* $\|\boldsymbol{x}_1\|^2 = 4\lambda^2\cos^2\frac{\theta_0}{2}$ *and*

$$\begin{aligned}
E''(0) &= \Psi(\theta_0)\left(4w_2\lambda^2\cos^2\frac{\theta_0}{2}(1 - A(\theta_0)) + 2w_3(1-\cos\theta_0)\right) \\
&\quad + (w_1 - w_2)\frac{8\theta_0}{\sin\theta_0}\left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0\cot\frac{\theta_0}{2}\right)\lambda^2\cos^4\frac{\theta_0}{2} \\
&= \frac{\lambda^2\left(5w_2 - 12w_1\right) + 15w_3}{90}\theta_0^4 + \frac{\lambda^2\left(216w_1 - 91w_2\right) - 189w_3}{7560}\theta_0^6 + O\left(\theta_0^8\right).
\end{aligned}$$

Hence, as in the isotropic case $E''(0)$ *is not necessarily positive, even when considering the restriction on the constants* w_1 *and* w_2*.*

Finally, we consider $\boldsymbol{x}_1$ *perpendicular to the previous case, with* $\boldsymbol{x}_1 = \lambda(\boldsymbol{n}_0 - \boldsymbol{n}_1)$*. Then we know that* $\cos\alpha_0 = 0$*, and* $\|\boldsymbol{x}_1\|^2 = 2\lambda^2(1-\cos\theta_0)$ *such that*

$$E''(0) = 2\Psi(\theta_0)\left(w_2\lambda^2 + 2w_3\right)(1-\cos\theta_0) > 0,$$

i.e. in this case we always have a minimum.

Next we consider the most general coplanar case: $\boldsymbol{x}_1 = \lambda\boldsymbol{n}_0 + \mu\boldsymbol{n}_1$ for arbitrary constants $\lambda, \mu \in \mathbb{R}$. Using Equation (5.7), this leads to the following theorem:

Theorem 5.2 (**Coplanar solutions to the minimisation problem in general**). *Consider the general optimisation problem, Equation (4.55) with* $\boldsymbol{x}_0 = \boldsymbol{0}$ *for the coplanar case (i.e. Eq. (4.48)) in the spatially anisotropic metric* $\mathcal{G}$ *as defined in Eq. (3.18): with* $w_1, w_2 \geqslant 0$ *under the restriction* $w_1 \leqslant w_2 \leqslant 4w_1$*.*

Then the maximally subset S *of end-points* $\boldsymbol{p}_1 \in \mathbb{M}_3$ *where*

$$\sigma_\rho|_S = \sigma_{\mathrm{MAV}}|_S \Leftrightarrow \varphi = 0 \text{ minimises } E(\varphi),$$

is given by

$$\boxed{\begin{aligned}
S &= \left\{\boldsymbol{p}_1 = (\boldsymbol{x}_1, \boldsymbol{n}_1) \in \mathbb{M}_3 \;\middle|\; \boldsymbol{x}_1 = \lambda\boldsymbol{n}_0 + \mu\boldsymbol{n}_1, \boldsymbol{n}_0\cdot\boldsymbol{n}_1 = \cos\theta_0, \widetilde{F}(\lambda,\mu,\theta_0) > 0\right\}, \\
\widetilde{F}(\lambda,\mu,\theta_0) &= 2w_3\Psi(\theta_0)(1-\cos\theta_0) \\
&\quad + w_2\Psi(\theta_0)\left((\lambda+\mu)^2\left(1 - \cos^2\tfrac{\theta_0}{2}A(\theta_0)\right) - 4\lambda\mu\sin^2\tfrac{\theta_0}{2}\right) \\
&\quad + 2(w_1 - w_2)(\lambda+\mu)^2\left(\tfrac{\sin\theta_0 - \theta_0}{\theta_0}\left(\tfrac{\theta_0}{2}\cot\tfrac{\theta_0}{2}\right)^2 + \Psi(\theta_0)\cos^2\tfrac{\theta_0}{2}\right),
\end{aligned}} \tag{5.14}$$

[1] Despite the introduced anisotropy, the only vector-dependence is still contained in α_0.

where the functions Ψ and A are defined in Equations (5.3a) and (5.3b) respectively.

Remark. *As in the isotropic metric, we can define the set S_0 as the set S in the case that $w_3 = 0$.*

Lemma 5.1. *In the previous Theorem 5.2, the function $\widetilde{F}$ in Equation (5.14) can be expressed in a Taylor series in θ_0 as*

$$\widetilde{F}(\lambda,\mu,\theta_0) = \left(\frac{\lambda^2+\mu^2-\lambda\mu}{18}w_2 - w_1\frac{(\lambda+\mu)^2}{30} + \frac{w_3}{6}\right)\theta_0^4 \tag{5.15a}$$

$$+\left(\frac{(\lambda+\mu)^2}{140}w_1 - \frac{\lambda^2-\frac{7}{10}\lambda\mu+\mu^2}{108}w_2 - \frac{w_3}{40}\right)\theta_0^6 + O\left(\theta_0^8\right). \tag{5.15b}$$

Remark. *This means that for small angles θ_0 (in which the first term of the series dominates) the extremum $\varphi = 0$ is a minimum (and thus section $\sigma_{\mathrm{MAV}} = \sigma_\rho$) as long as*

$$5w_2(\lambda^2+\mu^2-\lambda\mu) + 15w_3 > 3w_1(\lambda+\mu)^2. \tag{5.16}$$

For larger values of θ_0, the θ_0^6-term must be taken into account as well to determine whether this extremum is a minimum.

Remark. *In the anisotropic case of Theorem 5.2, we have that increasing w_3 will increase the size of the set S, as was the case in the isotropic case, Thm. 5.1. This will be discussed in Lemma 5.3. For $w_3 = 0$ the series in Equation (5.15) can be written in terms of the spatial anisotropy $\zeta = \frac{w_2}{w_1}$. This means that also Eq. (5.16) can be rewritten as*

$$\zeta > \frac{3}{5}\frac{(\lambda+\mu)^2}{\lambda^2+\mu^2-\lambda\mu}, \tag{5.17}$$

where the right hand side is bounded above by $\frac{12}{5}$. Hence, for $\zeta > \frac{12}{5}$ and θ_0 small enough, we have that $\varphi \mapsto E(\varphi)$ has its minimum at $\varphi = 0$ without any further restrictions on the position $\boldsymbol{x}_1$. This inequality is illustrated in Figure 5.2, where the shaded region is the region in which it is satisfied.

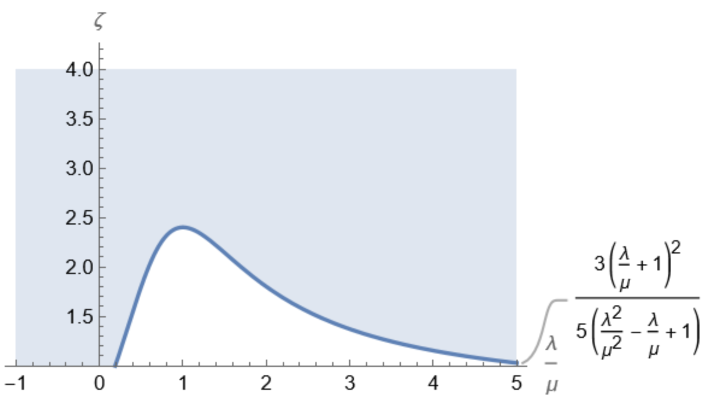


Figure 5.2: Plot of the inequality in Eq. (5.17) for ζ and ratio $\frac{\lambda}{\mu}$. The shaded area means that the inequality is satisfied, i.e. the extremum $E(0)$ is a minimum.

5.1.3 Properties and symmetries of the maximal set S on which $\sigma_\rho|_S = \sigma_{\mathrm{MAV}}|_S$

In general, i.e. in the settings of both Theorem 5.1 and the more general Theorem 5.2, we have the following lemmas about the results of scaling and rotations on the set S, as defined in either Theorem.

Lemma 5.2 (Scaling $\boldsymbol{x}_1$)**.** *If the position $\boldsymbol{x}_1$ is scaled by a factor $\nu > 0$, i.e. $\boldsymbol{x}_1 \mapsto \nu\boldsymbol{x}_1$, then:*

- *For $\nu \ll 1$: $\varphi \mapsto E(\varphi)$ has a minimum at $\varphi = 0$, i.e. $\nu\boldsymbol{x}_1 \in S$, for $w_3 > 0$,*
- *For $\nu \gg 1$: then $\nu\boldsymbol{x}_1 \in S$ if and only if $\boldsymbol{x}_1 \in S_0$.*

Proof. Multiply λ and μ by ν. Then in e.g. Eq. (5.14): for $\nu \ll 1$ the (always positive) w_3-term dominates for $w_3 > 0$, while for $\nu \gg 1$ the w_1 and w_2-terms dominate. □

Remark. *This means that for small $\|\boldsymbol{x}_1\|$, i.e. $\|\boldsymbol{x}_1\| < r$ for some $r > 0$, E will always have a minimum at $\varphi = 0$, while for large $\|\boldsymbol{x}_1\|$ this is only a minimum if it is a minimum for $w_3 = 0$.*

Lemma 5.3 (Scaling w_3)**.** *Increasing w_3 increases the size of the set S. It also increases the radius of the ball around the origin $\mathbf{0}$, see the previous remark, in which $\boldsymbol{x}_1$ can lie to guarantee minimising $E(\varphi)$ at $\varphi = 0$.*

Proof. The w_3-term is always positive, such that increasing w_3 will always increase the function $\widetilde{F}$, which grows the set S. For the second part of the statement we see that increasing w_3 allows it to dominate more for $\nu \ll 1$, following the previous proof. □

Lemma 5.4 (Rotational symmetry)**.** *The set $S \subset \mathbb{M}_3$ has the rotational symmetry*

$$h \triangleright S = S \tag{5.18}$$

for all $h \in H = \operatorname{stab} \boldsymbol{p}_0 \subset \mathrm{SE}(3)$, i.e. for all $\boldsymbol{p}_1 \in S$ we have that $h \triangleright \boldsymbol{p}_1 \in S$.

Proof. Recall or notice that such a rototranslation h consists of only a rotational component. Let $\boldsymbol{R}$ be any rotation around $\boldsymbol{n}_0$ and define

$$\widetilde{\boldsymbol{n}}_0 = \boldsymbol{R}\boldsymbol{n}_0 = \boldsymbol{n}_0, \quad \widetilde{\boldsymbol{n}}_1 = \boldsymbol{R}\boldsymbol{n}_1, \quad \widetilde{\boldsymbol{x}}_1 = \boldsymbol{R}\boldsymbol{x}_1 = \lambda\widetilde{\boldsymbol{n}}_0 + \mu\widetilde{\boldsymbol{n}}_1.$$

From this definition we see that λ and μ are conserved under this rotation, and the angle θ_0 between $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ is as well since

$$\cos\widetilde{\theta}_0 = \widetilde{\boldsymbol{n}}_0 \cdot \widetilde{\boldsymbol{n}}_1 = \boldsymbol{n}_0 \cdot \boldsymbol{R}^\top\boldsymbol{R}\boldsymbol{n}_1 = \boldsymbol{n}_0 \cdot \boldsymbol{n}_1 = \cos\theta_0.$$

Thus, under this rotation the function $\widetilde{F}$ defining S does not change, and S is symmetric under such rotations. □

5.2 The general case

In this section we discuss the case that the points $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are not necessarily coplanar, in which case Eq. (4.48) does not vanish. Then, we do not have the guarantee of Theorem 4.1 that the extremum of $E(\varphi)$ will remain at $\varphi = 0$. Instead we find that this may move around to some $\varphi^* \neq 0$. Hence, we attempt to find this minimiser.

We do not have an explicit formula for $E'(\varphi)$ that we can solve directly, but as we do know the first and second derivatives of E at $\varphi = 0$ (see Theorems 4.2 and 4.3), we may simply give a Taylor series expansion of E around $\varphi = 0$ and hope that the minimiser φ_* of the truncated expansion is a decent approximation of the minimiser of E.

Lemma 5.5 **(3rd order Taylor expansion of E).**
We can write E in a third order Taylor series around $\varphi = 0$ as

$$E(\varphi) = \sum_{k=0}^{3} \frac{E^{(k)}}{k!}\varphi^k + O(\varphi^4) = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + O(\varphi^4),$$

with coefficients

$$\boxed{\begin{aligned}
a_0 &= w_2\left(\frac{\|\boldsymbol{x}_1\|^2}{\operatorname{sinc}^2\frac{\theta_0}{2}} - \frac{|\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2}{2}A(\theta_0)\Psi(\theta_0)\right) + (w_1 - w_2)\left|\frac{\boldsymbol{x}_1\cdot(\boldsymbol{n}_0+\boldsymbol{n}_1)}{2\operatorname{sinc}\theta_0}\right|^2 + \theta_0^2 w_3,\\
a_1 &= (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})\left((w_1 - w_2)\cot\tfrac{\theta_0}{2}\tfrac{\sin\theta_0-\theta_0}{\operatorname{sinc}\theta_0} - w_2 A(\theta_0)\Psi(\theta_0)\right),\\
a_2 &= w_2 A(\theta_0)\Psi(\theta_0)\left(|\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2 - \left|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right|^2\right)\\
&\quad+\Psi(\theta_0)\left(w_2\left(\|\boldsymbol{x}_1\|^2 - |\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2\right) + 2w_3(1-\cos\theta_0)\right)\\
&\quad+(w_1 - w_2)\left(\theta_0\cot\tfrac{\theta_0}{2}\left(3 - \tfrac{\theta_0}{\sin\theta_0} - \theta_0\cot\tfrac{\theta_0}{2}\right)\left|\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right|^2 + |\boldsymbol{x}_1\cdot\boldsymbol{k}_0|^2\tfrac{(\sin\theta_0-\theta_0)^2}{2\sin^2\frac{\theta_0}{2}}\right),\\
a_3 &= (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})w_1\cot\tfrac{\theta_0}{2}\tfrac{10\theta_0+\frac{9\sin^2\theta_0}{\theta_0}-22\sin\theta_0-\frac{3}{2}\sin 2\theta_0+6\theta_0\cos\theta_0}{6\operatorname{sinc}\theta_0}\\
&\quad+(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})\tfrac{w_2}{6}\cos^2\tfrac{\theta_0}{2}\tfrac{2\left(3\theta_0\sin\theta_0-3\theta_0^2+8\right)\cos\theta_0+32\theta_0\sin\theta_0+9\cos 2\theta_0-6\theta_0^2-25}{\sin^2\theta_0}.
\end{aligned}} \tag{5.19}$$

where the functions Ψ and A are defined in Equations (5.3a) and (5.3b) respectively.

We approximate E with the truncated series, i.e.

$$E(\varphi) \approx a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 =: E^{\{3\}}(\varphi). \tag{5.20}$$

Proof. Recall that the function E is given by Equation (4.56) and its first and second order derivatives at $\varphi = 0$ are in the most general metric (so possibly anisotropic) given by Theorem 4.3, Equations (4.59) and (4.60). The other coefficients a_0 and a_3 can be found by computing $E(0)$ and $E^{(3)}(0)$ in a similar manner. ☐

Remark. *Notice that, as was given in Lemma 4.2, the odd coefficients a_1 and a_3 contain the product $(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})$.*

Remark. *It is also possible to see that a_2 vanishes in the limit $\theta_0 \to 0$.*

Remark. *The function E and its 3rd order Taylor approximation $E^{\{3\}}$, Eq. (5.20), are depicted in Fig. 5.3.*

Thus, we approximate the minimisers of E by the critical points of $E^{\{3\}}$[2], which are given by

$$\varphi_\pm = \frac{-a_2 \pm \sqrt{a_2^2 - 3a_1a_3}}{3a_3}. \tag{5.21}$$

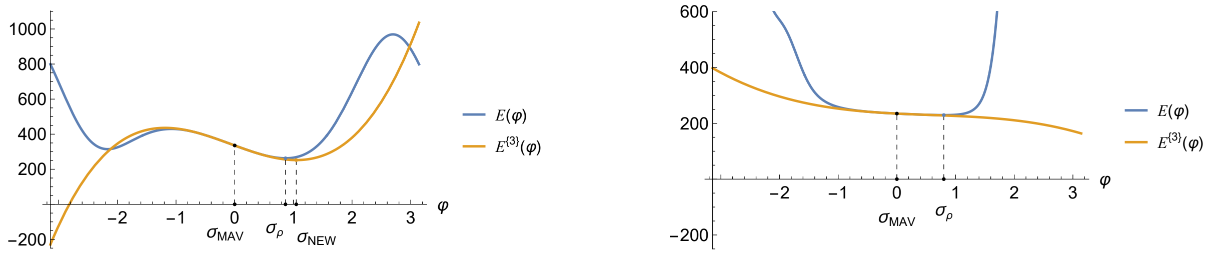
Remark. *Notice that these solutions are only real when*

$$a_2^2 \geqslant 3a_1a_3. \tag{5.22}$$

Notice also that in the coplanar case (and in mirror) Equation (5.21) does not directly exist due to the division by a_3, even though the previous condition is satisfied. In this case the problem is only in applying the quadratic formula to a linear equation, and the true critical point is at $\varphi = 0$. This is the subject of Thm. 5.4.

As this is a third order approximation, we know that the pair of solutions $\varphi_\pm$ must consist of both a minimiser and a maximiser. This is the subject of the following Theorem.

[2]As we are interested only in the minimiser close to $\varphi = 0$, we only consider the critical points and not all minimisers.



(a) With real minimiser φ_+

(b) With imaginary minimiser φ_+

Figure 5.3: Plot of the function E and its third order Taylor series $E^{\{3\}}$, with (the values of φ corresponding to) the section $\sigma_{\text{MAV}}, \sigma_\rho$ and σ_{NEW}, see Definition 5.1.

Theorem 5.3. *When approximating the function E as given in Equation (4.56) with the truncated series $E^{\{3\}}$ as in Lemma 5.5, under the condition that Equation (5.22) is satisfied, the minimiser φ_* of $E^{\{3\}}$ is given by $\varphi_+ \in \mathbb{R}$ in Equation (5.21), i.e. the minimiser is*

$$\varphi_* = \varphi_+ = \frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3} \tag{5.23}$$

for $a_3 \neq 0$.

Remark. *For $a_3 = 0$, the critical point is given by $\varphi_* = -\frac{a_1}{2a_2}$, which is a minimiser only when $a_2 > 0$. We use the same symbol φ_* to refer to the minimiser in both situations.*

Proof. By taking the second derivative of Equation (5.20) and substituting $\varphi_\pm$, we see that

$$\begin{aligned}(E^{\{3\}})''(\varphi_\pm) &= 2a_2 + 6a_3\varphi_\pm \\ &= 2a_2 + 2\left(-a_2 \pm \sqrt{a_2^2 - 3a_1a_3}\right) = \pm 2\sqrt{a_2^2 - 3a_1a_3},\end{aligned}$$

such that by the second derivative test, the positive sign yields the minimiser. □

As we have defined the parameter φ to lie in the interval $[-\pi, \pi)$, we are interested in an upper bound of $|\varphi_*|$ to ensure that this lies in the correct interval. We have the following Lemma on approximation and upper bound of $|\varphi_*|$

Lemma 5.6. *When considering the minimiser φ_* as given in Theorem 5.3, its absolute value has the following general upper bound*

$$|\varphi_*| \leqslant \left|\frac{a_1}{a_2}\right|, \tag{5.24}$$

with equality only when a_1 or a_3 vanishes, or when the inequality in Eq. (5.22) is an equality.

Furthermore, when $a_1a_3 \ll a_2^2$, we have the approximation

$$|\varphi_*| \approx \frac{1}{2}\left|\frac{a_1}{a_2}\right|. \tag{5.25}$$

Proof. In the $a_3 = 0$ case, we see that the approximation is exact. In the other case we see that the absolute value of the minimiser φ_+ can be expressed as

$$|\varphi_+| = \left|\frac{-a_2 + a_2\sqrt{1 - 3\frac{a_1a_3}{a_2^2}}}{3a_3}\right| = \frac{1}{3}\left|\frac{a_2}{a_3}\right|\left|\sqrt{1 - 3\frac{a_1a_3}{a_2^2}} - 1\right|.$$

The function $|\sqrt{1-x}-1|$ has a first order approximation (in x) given by $\frac{1}{2}|x|$, and upper bound $|\sqrt{1-x}-1| \leqslant |x|$ for all $x \in (-\infty, 1]$ with equality only when $x = 0$ or $x = 1$, proving the required result. $\square$

Remark. *While this is an upper bound, it does not allow us to confirm whether $|\varphi_*| \leqslant \pi$. Further simplification is hindered by the complicated form of the coefficient a_2.*

Due to the nature of this expansion, we know that when the true minimiser of E is close to $\varphi = 0$, the approximate minimiser φ_* captures this behaviour well, see for example the situation in Figure 5.3. When the true minimiser lies farther from $\varphi = 0$, the approximate minimiser φ_*, while less accurate than the case above, is expected to be more accurate than the simple minimiser $\varphi = 0$, such that in both cases, the approximate minimiser φ_* is a better approximation than the simple approximation $\varphi = 0$ of the MAV section and generator. Hence, we can use this to define the following new and more accurate approximation.

Definition 5.1 (Section σ_{NEW}). We define the new approximation section σ_{NEW} as the *partial*[3] section that returns the generator with $\varphi = \varphi_*$ as given in Theorem 5.3 and the following remark whenever this is real:

$$\sigma_{\text{NEW}}([g]) = \exp(\chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\varphi_*)), \tag{5.26}$$

where $[g] \in G/H$ is the equivalence class of rototranslations mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$.

Remark. *It is possible (with a little more effort) to also compute the coefficient a_4 and possibly even further in order to define better approximations. This is however not recommended, as the derivative would lead to a cubic (or higher order) equation whose solutions are much less nice to write down than those of the quadratic in Equation (5.21).*

Next we show that the new section σ_{NEW} coincides with the old section σ_{MAV} characterised by $\varphi = 0$ in the case of coplanar end-conditions.

Theorem 5.4 (Convergence to the coplanar case: σ_{NEW} extends σ_{MAV}). *Let $\boldsymbol{x}_0 = \boldsymbol{0}$ and $C := \det(\boldsymbol{x}_1 - \boldsymbol{x}_0 \mid \boldsymbol{n}_0 \mid \boldsymbol{n}_1) = (\boldsymbol{x}_1 - \boldsymbol{x}_0) \cdot (\boldsymbol{n}_0 \times \boldsymbol{n}_1) = \boldsymbol{x}_1 \cdot \boldsymbol{k}_0$ be the coplanarity condition as defined in Def. 3.2. Then if $C = 0$, i.e. we are in the coplanar case, we have that the approximate minimiser φ_* on which the new section σ_{NEW}, Def. 5.1, depends vanishes, i.e. $\varphi_* = 0$ and thereby the new section σ_{NEW} coincides with the section σ_{MAV} in the coplanar case and is a perfect approximation of σ_ρ when σ_{MAV} is, i.e. $\sigma_{\text{NEW}}|_S = \sigma_{\text{MAV}}|_S = \sigma_\rho|_S$, where S is given in Theorem 5.1 and 5.2.*

Proof. From the remark under Thm. 5.3, we see that in the coplanar case, i.e. when the odd coefficients a_1, a_3 vanish, the critical point is given by $\varphi_* = 0$, which is a minimiser only when $a_2 > 0$, which we know to be the case on the set S, as there $\varphi = 0$ is a minimiser (see Thm. 5.1 and 5.2). $\square$

As the definition of the new section σ_{NEW}, Def. 5.1, is conditional on the existence of the minimiser φ_*, we may also define another new section that is given by σ_{NEW} whenever this exists, and by a continuous extension otherwise. About this section, we make the following claim:

Theorem 5.5 (The newly proposed section $\overline{\sigma}_{\text{NEW}}$ that better approximates σ_ρ). *Set $G = \text{SE}(3)$, $H = \text{stab}_G(\boldsymbol{p}_0) \equiv \text{SO}(2)$. Then the section $\overline{\sigma}_{\text{NEW}} : G/H \to G$ given by*

$$\overline{\sigma}_{\text{NEW}}([g]) = \exp\left(\chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\overline{\varphi})\right),$$

[3]it is only defined on points $\boldsymbol{p}_1$ such that $\varphi_* = \varphi_*(\boldsymbol{p}_0, \boldsymbol{p}_1)$ exists, i.e. $a_2^2 \geqslant 3a_1a_3$

where $[g] \in G/H$ is the equivalence class of rototranslations mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}_1$, $\chi_{\boldsymbol{p}_0,\boldsymbol{p}_1}(\overline{\varphi}) = \chi(\overline{\varphi})$ given by Eq. (4.39), and with

$$\overline{\varphi} = \begin{cases} -\frac{a_2}{3a_3}, & \text{if } a_2^2 < 3a_1a_3 \\ \frac{-a_2+\sqrt{a_2^2-3a_1a_3}}{3a_3}, & \text{if } a_2^2 \geqslant 3a_1a_3 \text{ and } a_3 \neq 0 \\ -\frac{a_1}{2a_2}, & \text{if } a_3 = 0 \text{ and } a_2 \neq 0 \\ 0, & \text{if both } a_2 = 0 \text{ and } a_3 = 0 \end{cases}, \tag{5.27}$$

that extends the partial section σ_{NEW} is a bounded continuous section, in the sense that as $a_2^2 - 3a_1a_3 \to 0$, we have that $\varphi_* \to -\frac{a_2}{3a_3}$. Again on the set S (as given in Thm. 5.1 and 5.2) it is exact: $\overline{\sigma}_{\text{NEW}}|_S = \sigma_\rho|_S$. It better approximates σ_ρ outside S in the sense that it relies on higher order Taylor approximation in φ (that holds also outside the coplanar case).

Remark. *Notice that the middle two cases correspond to φ_*, i.e. the partial section σ_{NEW}, the top case is the continuous extension in the case that Eq. (5.22) is not satisfied, and the bottom case is the final remaining edge case.*

Proof. Regarding continuity: This follows from the fact that $\lim_{a_2^2-3a_1a_3\downarrow 0} \varphi_* = -\frac{a_2}{3a_3}$.

Regarding the exactness statement on S: This follows from the fact that in the coplanar case this reduces to $\overline{\varphi} = 0$ and Thms. 5.1 and 5.2.

Regarding boundedness: For φ_* this boundedness was the subject of Lemma 5.6 and for the continuous extension one has, under condition $a_2^2 < 3a_1a_3$,

$$\left| -\frac{a_2}{3a_3} \right| \leqslant \sqrt{\left| \frac{a_1}{3a_3} \right|} = f(\theta_0), \tag{5.28}$$

where the ratio of a_1 and a_3 is a function only of θ_0 and metric coefficients w_1, w_2. Under the condition that $a_2^2 < 3a_1a_3$, this is a bounded continuous function $f(\theta_0)$ (without singularities), see also Figure 5.4

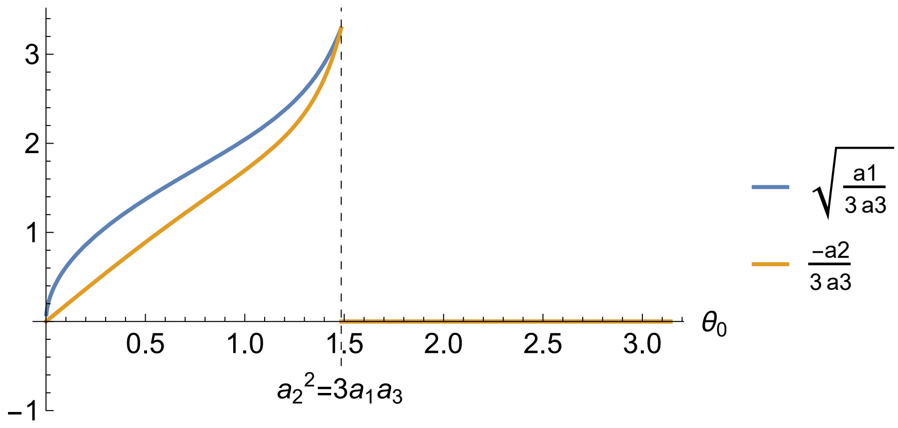


Figure 5.4: Visualisation of the upper bound of Eq. (5.28), plotted as the piecewise functions $\left|-\frac{a_2}{3a_3}\right|$ and $\sqrt{\left|\frac{a_1}{3a_3}\right|}$ whenever $a_2^2 < 3a_1a_3$ and 0 otherwise. This also shows the boundedness and continuity of the function f for at least this particular choice of $\boldsymbol{n}_0$ and $(\boldsymbol{x}_1, \boldsymbol{n}_1)$.

Note that φ_* follows from a 3rd-order series expansion (Eq. (5.20)) that holds for any $\boldsymbol{p}_1 \in G/H$ (also outside of S), while the approximation $\overline{\varphi} = 0$ follows from a 2nd order series that produces a minimiser only when $\boldsymbol{p}_1 \in S$. $\square$

Chapter 6

Conclusion

In this thesis we have investigated the error between the section σ_{MAV} and σ_ρ in Theorem 3 of [16] by analysing the auxiliary function $E(\varphi) = \|\chi(\varphi) \triangleright \boldsymbol{p}_0\|^2_{\mathcal{G}} = \rho_{\widetilde{\mathcal{G}}}(e^{\chi(\varphi)}, e)^2$, using a new (screw-motion) parametrisation of all generators $\chi(\varphi)$ mapping $\boldsymbol{p}_0$ to $\boldsymbol{p}_1 = e^{\chi(\varphi)} \triangleright \boldsymbol{p}_0$. This parametrisation turned out to be highly preferable over standard Euler angle parametrisation and even canonical log-coordinates parametrisations (as studied in previous work, e.g. [8, 16]), in the overall problem of finding the elements $g = e^{\chi(\varphi_{\mathrm{opt}})}$ in the fibres $[g]$ of $\mathbb{M}_3 = G/H$, $G = \mathrm{SE}(3), H = \mathrm{stab}_G(\boldsymbol{p}_0)$ with minimal logarithmic norm.

In the case that $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ are coplanar, we have seen that $\varphi = 0$ is always an extremum of E (Thm. 4.1), i.e. σ_{MAV} always provides an extremum of $\rho_{\mathcal{G}}$. When this extremum is a minimum we have that $\sigma_{\mathrm{MAV}} = \sigma_\rho$. In Theorems 5.1 and 5.2 we successfully derived the set $S \subset \mathbb{M}_3$ on which this extremum is a minimum for spatially (an)isotropic metrics, as well as several further Lemmas about this set.

In the general case where $\boldsymbol{p}_0$ and $\boldsymbol{p}_1$ need not be coplanar, we expand $E(\varphi)$ as a third order Taylor series in φ (Lemma 5.5) in order to provide an approximate minimiser. This minimiser is given by Theorem 5.3 and leads to the partial section σ_{NEW} as defined in Definition 5.1 and its extension to a full section $\overline{\sigma}_{\mathrm{NEW}}$ as discussed in Theorem 5.5. These sections provide a better approximation of σ_ρ than σ_{MAV}. Furthermore, by Theorem 5.4 and 5.5 they converge to the correct coplanar solution $\varphi = 0$ if the end conditions tend to a coplanar configuration, i.e. they coincide with σ_{MAV}, which on the set S from Thms. 5.1 and 5.2 provides the true minimiser σ_ρ of ρ: $\overline{\sigma}_{\mathrm{NEW}}|_S = \sigma_{\mathrm{MAV}}|_S = \sigma_\rho|_S$.

Finding the exact minimiser of E seems impossible given the complexity of the function.

It may be possible to generalise this screw motion idea to that of a (more) general homogeneous space G/H in a Lie group $G = V \rtimes H$ acting on $V \times W$ by first considering the action of H on W. For this latter idea we mention that Chasles' theorem, which allows screw motion to occur, is a special case of invariant decomposition.

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Appendix A

Differentiating the norm

A.1 Differentiating $\|\dot{\boldsymbol{n}}\|^2$

We are given a (semi-)explicit formula for $\dot{\boldsymbol{n}}$, see Equation (4.47b), such that we can simply compute

$$\begin{aligned}
\|\dot{\boldsymbol{n}}\|^2 &= \theta_\varphi^2 \left(1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi\right) \\
\frac{\partial}{\partial\varphi}\|\dot{\boldsymbol{n}}\|^2 &= 2\theta_\varphi\theta'_\varphi \left(1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi\right) - \theta_\varphi^2\left(\cos^2\frac{\theta_0}{2}\sin 2\varphi\right) \\
\frac{\partial^2}{\partial\varphi^2}\|\dot{\boldsymbol{n}}\|^2 &= 2\left(\theta_\varphi\theta''_\varphi + (\theta'_\varphi)^2\right)\left(1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi\right) - 2\theta_\varphi^2\left(\cos^2\frac{\theta_0}{2}\cos 2\varphi\right) \\
&\quad - 4\theta_\varphi\theta'_\varphi\left(\cos^2\frac{\theta_0}{2}\sin 2\varphi\right).
\end{aligned}$$

Using the definition of θ_φ, Eqns. (4.31) and (4.32), we can also see that for φ close to zero, we have

$$\theta'_\varphi = \sin\varphi \frac{\sin\theta_0}{1 - \cos^2\frac{\theta_0}{2}\sin^2\varphi} \qquad \Longrightarrow \theta'_0 = 0, \tag{A.1}$$

$$\theta''_\varphi = \frac{\sin\theta_0\cos\varphi\left(1 + \cos^2\frac{\theta_0}{2}\sin^2\varphi\right)}{\left(\cos^2\frac{\theta_0}{2}\sin^2\varphi - 1\right)^2} \quad \Longrightarrow \theta''_0 = \sin\theta_0. \tag{A.2}$$

Hence, we have

$$\left.\frac{\partial}{\partial\varphi}\|\dot{\boldsymbol{n}}\|^2\right|_{\varphi=0} = 0 \quad \text{and} \quad \left.\frac{\partial^2}{\partial\varphi^2}\|\dot{\boldsymbol{n}}\|^2\right|_{\varphi=0} = 2\theta_0\left(\sin\theta_0 - \theta_0\cos^2\frac{\theta_0}{2}\right). \tag{A.3}$$

A.2 Differentiating $\|\dot{\boldsymbol{x}}\|^2$

We are not given an explicit formula for $\|\dot{\boldsymbol{x}}\|^2$ yet, such that we have to construct this from Equation (4.47a). Here we have

$$\begin{aligned}
\|\dot{\boldsymbol{x}}\|^2 &= (\boldsymbol{x}_\parallel + \boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi)\cdot(\boldsymbol{x}_\parallel + \boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi) \\
&= \boldsymbol{x}_\parallel\cdot\boldsymbol{x}_\parallel + (\boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi)\cdot(\boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi) + 2\boldsymbol{x}_\parallel\cdot(\boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi), \\
&= (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)^2\cancelto{1}{\boldsymbol{k}_\varphi\cdot\boldsymbol{k}_\varphi} + \theta_\varphi^2(\|\boldsymbol{c}\|^2 - (\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)^2) + 0.
\end{aligned} \tag{A.4}$$

This means that

$$\frac{\partial}{\partial\varphi}\|\dot{\boldsymbol{x}}\|^2 = \theta_\varphi^2\left(\frac{\partial}{\partial\varphi}\|\boldsymbol{c}(\varphi)\|^2 - 2(\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)(\boldsymbol{x}_m\cdot\boldsymbol{k}'_\varphi)\right) + 2\theta_\varphi\theta'_\varphi\left(\|\boldsymbol{c}(\varphi)\|^2 - (\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)^2\right)$$
$$+ 2(\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi),(\boldsymbol{x}_1\cdot\boldsymbol{k}'_\varphi)$$
$$\frac{\partial^2}{\partial\varphi^2}\|\dot{\boldsymbol{x}}\|^2 = \theta_\varphi^2\left(\frac{\partial^2}{\partial\varphi^2}\|\boldsymbol{c}(\varphi)\|^2 - 2(\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)(\boldsymbol{x}_m\cdot\boldsymbol{k}''_\varphi) - 2\left(\boldsymbol{x}_m\cdot\boldsymbol{k}'_\varphi\right)^2\right)$$
$$+ 4\theta_\varphi\theta'_\varphi\left(\frac{\partial}{\partial\varphi}\|\boldsymbol{c}(\varphi)\|^2 - 2(\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)(\boldsymbol{x}_m\cdot\boldsymbol{k}'_\varphi)\right)$$
$$+ 2\left(\theta_\varphi\theta''_\varphi + (\theta'_\varphi)^2\right)\left(\|\boldsymbol{c}(\varphi)\|^2 - (\boldsymbol{x}_m\cdot\boldsymbol{k}_\varphi)^2\right) + 2(\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)(\boldsymbol{x}_1\cdot\boldsymbol{k}''_\varphi) + 2\left(\boldsymbol{x}_1\cdot\boldsymbol{k}'_\varphi\right)^2,$$

where for the derivatives we have that $\boldsymbol{k}'_\varphi = \boldsymbol{k}_{\varphi+\frac{\pi}{2}}$ and $\boldsymbol{k}''_\varphi = -\boldsymbol{k}_\varphi$, from its definition in Equation (4.24), and we also have that (Equation (4.38))

$$\boldsymbol{c}(\varphi) = \boldsymbol{x}_m + \frac{1}{4}\cot\frac{\theta_0}{2}\left((1+\cos 2\varphi)(\boldsymbol{k}_0\times\boldsymbol{x}_1) + \sin 2\varphi(\boldsymbol{k}_{\frac{\pi}{2}}\times\boldsymbol{x}_1)\right), \tag{A.5}$$
$$\boldsymbol{c}'(\varphi) = \frac{1}{2}\cot\frac{\theta_0}{2}\left(-\sin 2\varphi(\boldsymbol{k}_0\times\boldsymbol{x}_1) + \cos 2\varphi(\boldsymbol{k}_{\frac{\pi}{2}}\times\boldsymbol{x}_1)\right),$$
$$\boldsymbol{c}''(\varphi) = -\cot\frac{\theta_0}{2}(\cos 2\varphi(\boldsymbol{k}_0\times\boldsymbol{x}_1) + \sin 2\varphi(\boldsymbol{k}_{\frac{\pi}{2}}\times\boldsymbol{x}_1)),$$

such that

$$\|\boldsymbol{c}(0)\|^2 = \boldsymbol{c}(0)\cdot\boldsymbol{c}(0) = \frac{1}{2}\frac{\|\boldsymbol{x}_1\|^2}{1-\cos\theta_0} - \frac{1}{4}\cot^2\frac{\theta_0}{2}(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)^2$$
$$\left.\frac{\partial}{\partial\varphi}\|\boldsymbol{c}(\varphi)\|^2\right|_{\varphi=0} = 2\boldsymbol{c}(0)\cdot\boldsymbol{c}'(0) = -\frac{1}{2}\cot^2\frac{\theta_0}{2}(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})$$
$$\left.\frac{\partial^2}{\partial\varphi^2}\|\boldsymbol{c}(\varphi)\|^2\right|_{\varphi=0} = 2(\|\boldsymbol{c}'(0)\|^2 + \boldsymbol{c}(0)\cdot\boldsymbol{c}''(0)) = \frac{1}{2}\cot^2\frac{\theta_0}{2}\left(\|\boldsymbol{k}_{\frac{\pi}{2}}\times\boldsymbol{x}_1\|^2 - 2\|\boldsymbol{k}_0\times\boldsymbol{x}_1\|^2\right)$$
$$= \frac{1}{2}\cot^2\frac{\theta_0}{2}\left(2(\boldsymbol{x}_1\cdot\boldsymbol{k}_0)^2 - (\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})^2 - \|\boldsymbol{x}_1\|^2\right).$$

Substituting these derivatives, we have that

$$\left.\frac{\partial}{\partial\varphi}\|\dot{\boldsymbol{x}}\|^2\right|_{\varphi=0} = (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})\left(2 - \frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}}\right), \tag{A.6}$$
$$\left.\frac{\partial^2}{\partial\varphi^2}\|\dot{\boldsymbol{x}}\|^2\right|_{\varphi=0} = \left(\frac{\theta_0^2}{2\sin^2\frac{\theta_0}{2}} - 2\right)\left((\boldsymbol{x}_1\cdot\boldsymbol{k}_0)^2 - \left(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right)^2\right) \tag{A.7}$$
$$+ \theta_0\cot\frac{\theta_0}{2}\left(1 - \frac{\theta_0}{2}\cot\frac{\theta_0}{2}\right)\left(\|\boldsymbol{x}_1\|^2 - (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)^2\right).$$

A.3 Differentiating $|\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0|^2$

For the inner product we know that, see Equations (4.37) and (4.47a)

$$\dot{\boldsymbol{x}} = (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi + \boldsymbol{c}\times\theta_\varphi\boldsymbol{k}_\varphi \tag{A.8}$$
$$= \theta_\varphi\left[\boldsymbol{x}_m\times\boldsymbol{k}_\varphi + \frac{1}{2}\cot\frac{\theta_0}{2}|\cos\varphi|\boldsymbol{x}_1\right] + (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi\left[1 - \frac{\theta_\varphi}{2}\cot\frac{\theta_0}{2}|\cos\varphi|\right], \tag{A.9}$$

such that for $|\varphi| < \frac{\pi}{2}$

$$\begin{aligned}\frac{\partial}{\partial\varphi}\dot{\boldsymbol{x}} = \theta'_\varphi &\left[\boldsymbol{x}_m \times \boldsymbol{k}_\varphi + \frac{1}{2}\cot\frac{\theta_0}{2}\cos\varphi\boldsymbol{x}_1\right] + \theta_\varphi\left[\boldsymbol{x}_m \times \boldsymbol{k}'_\varphi - \frac{1}{2}\cot\frac{\theta_0}{2}\sin\varphi\boldsymbol{x}_1\right] \\ &+ ((\boldsymbol{x}_1\cdot\boldsymbol{k}'_\varphi)\boldsymbol{k}_\varphi + (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}'_\varphi)\left[1 - \frac{\theta_\varphi}{2}\cos\varphi\cot\frac{\theta_0}{2}\right] \\ &+ \frac{1}{2}(\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi\cot\frac{\theta_0}{2}(\sin\varphi\theta_\varphi - \cos\varphi\theta'_\varphi),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\varphi^2}\dot{\boldsymbol{x}} = (\theta''_\varphi - \theta_\varphi) &\left[\boldsymbol{x}_m \times \boldsymbol{k}_\varphi + \frac{1}{2}\cot\frac{\theta_0}{2}\cos\varphi\boldsymbol{x}_1\right] + 2\theta'_\varphi\left[\boldsymbol{x}_m \times \boldsymbol{k}'_\varphi - \frac{1}{2}\cot\frac{\theta_0}{2}\sin\varphi\boldsymbol{x}_1\right] \\ &+ 2((\boldsymbol{x}_1\cdot\boldsymbol{k}'_\varphi)\boldsymbol{k}'_\varphi - (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi)\left[1 - \cos\varphi\frac{\theta_\varphi}{2}\cot\frac{\theta_0}{2}\right] \\ &+ ((\boldsymbol{x}_1\cdot\boldsymbol{k}'_\varphi)\boldsymbol{k}_\varphi + (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}'_\varphi)\cot\frac{\theta_0}{2}(\sin\varphi\theta_\varphi - \cos\varphi\theta'_\varphi) \\ &- (\boldsymbol{x}_1\cdot\boldsymbol{k}_\varphi)\boldsymbol{k}_\varphi\cot\frac{\theta_0}{2}\left((\theta''_\varphi - \theta_\varphi)\cos\varphi - 2\theta'_\varphi\sin\varphi\right).\end{aligned}$$

Upon taking the inner product with $\boldsymbol{n}_0$ and setting $\varphi = 0$, we get

$$\begin{aligned}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0|_{\varphi=0} &= \frac{\theta_0}{2}\boldsymbol{x}_1\cdot\frac{\boldsymbol{n}_0+\boldsymbol{n}_1}{\sin\theta_0}, \\ \frac{\partial}{\partial\varphi}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0\bigg|_{\varphi=0} &= (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)\frac{\sin\theta_0 - \theta_0}{2\sin\frac{\theta_0}{2}}, \\ \frac{\partial^2}{\partial\varphi^2}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0\bigg|_{\varphi=0} &= \frac{1}{2}\left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0\cot\frac{\theta_0}{2}\right)\boldsymbol{x}_1\cdot(\boldsymbol{n}_0+\boldsymbol{n}_1).\end{aligned}$$

For the total derivatives this means that

$$\frac{\partial}{\partial\varphi}(\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0)^2\bigg|_{\varphi=0} = 2(\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0|_{\varphi=0})\left(\frac{\partial}{\partial\varphi}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0\bigg|_{\varphi=0}\right) = (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}})\cot\frac{\theta_0}{2}\frac{\sin\theta_0 - \theta_0}{\operatorname{sinc}\theta_0}, \tag{A.10}$$

$$\begin{aligned}\frac{\partial^2}{\partial\varphi^2}(\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0)^2\bigg|_{\varphi=0} &= 2\left((\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0|_{\varphi=0})\left(\frac{\partial^2}{\partial\varphi^2}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0\bigg|_{\varphi=0}\right) + \left(\frac{\partial}{\partial\varphi}\dot{\boldsymbol{x}}\cdot\boldsymbol{n}_0\bigg|_{\varphi=0}\right)^2\right) \\ &= \theta_0\cot\frac{\theta_0}{2}\left(3 - \frac{\theta_0}{\sin\theta_0} - \theta_0\cot\frac{\theta_0}{2}\right)\left(\boldsymbol{x}_1\cdot\boldsymbol{k}_{\frac{\pi}{2}}\right)^2 + (\boldsymbol{x}_1\cdot\boldsymbol{k}_0)^2\frac{(\sin\theta_0 - \theta_0)^2}{2\sin^2\frac{\theta_0}{2}}.\end{aligned} \tag{A.11}$$

Appendix B

Coplanarity: Finding λ and μ in the coplanar case

By Definition 3.2, we can write any coplanar pair of elements $(\boldsymbol{x}_0, \boldsymbol{n}_0), (\boldsymbol{x}_1, \boldsymbol{n}_1) \in \mathbb{M}_3$ as

$$\boldsymbol{x}_1 - \boldsymbol{x}_0 = \lambda \boldsymbol{n}_0 + \mu \boldsymbol{n}_1, \tag{B.1}$$

but λ and μ are not necessarily easy to find, as the two orientations $\boldsymbol{n}_0$ and $\boldsymbol{n}_1$ are not always orthogonal. In this appendix, where we make the simplification that $\boldsymbol{x}_0 = \boldsymbol{0}$, we offer two ways of finding these constants, both based on the following observation that

$$\boldsymbol{x}_1 \cdot \boldsymbol{n}_0 = \lambda + \mu \cos\theta_0 \tag{B.2}$$

$$\boldsymbol{x}_1 \cdot \boldsymbol{n}_1 = \mu + \lambda \cos\theta_0, \tag{B.3}$$

where $\cos\theta_0 = \boldsymbol{n}_0 \cdot \boldsymbol{n}_1$. Then we can find that

$$\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 + \boldsymbol{n}_1) = (\lambda + \mu)(1 + \cos\theta_0) \tag{B.4}$$

$$\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 - \boldsymbol{n}_1) = (\lambda - \mu)(1 - \cos\theta_0), \tag{B.5}$$

such that

$$\lambda = \frac{1}{2}\left(\frac{\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 + \boldsymbol{n}_1)}{1 + \cos\theta_0} + \frac{\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 - \boldsymbol{n}_1)}{1 - \cos\theta_0}\right) \tag{B.6}$$

$$\mu = \frac{1}{2}\left(\frac{\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 + \boldsymbol{n}_1)}{1 + \cos\theta_0} - \frac{\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 - \boldsymbol{n}_1)}{1 - \cos\theta_0}\right). \tag{B.7}$$

On the other hand, we can also notice that

$$\boldsymbol{x}_1 \cdot \boldsymbol{n}_0 \cos\theta_0 = \lambda \cos\theta_0 + \mu \cos^2\theta_0 \quad \text{and} \tag{B.8}$$

$$\boldsymbol{x}_1 \cdot \boldsymbol{n}_1 \cos\theta_0 = \mu \cos\theta_0 + \lambda \cos^2\theta_0, \tag{B.9}$$

such that

$$\boldsymbol{x}_1 \cdot (\boldsymbol{n}_1 \times (\boldsymbol{n}_0 \times \boldsymbol{n}_1)) = \boldsymbol{x}_1 \cdot \boldsymbol{n}_0 - \boldsymbol{x}_1 \cdot \boldsymbol{n}_1 \cos\theta_0 = \lambda \sin^2\theta_0 \tag{B.10}$$

$$\boldsymbol{x}_1 \cdot (\boldsymbol{n}_0 \times (\boldsymbol{n}_1 \times \boldsymbol{n}_0)) = \boldsymbol{x}_1 \cdot \boldsymbol{n}_1 - \boldsymbol{x}_1 \cdot \boldsymbol{n}_0 \cos\theta_0 = \mu \sin^2\theta_0, \tag{B.11}$$

where θ_0 is of course known.